Mathematics 150: Function Composition & Chain Rule

Many functions can be written as the composition of two (or more) functions that are somehow less complex than the original function. For example, the function R defined by $R(x) = \sqrt{4-9x}$ can be decomposed as

$$R(x) = h(f(x)) = (h \circ f)(x) \quad \text{where} \quad f(t) = 4 - 9t \quad \text{and} \quad h(t) = \sqrt{t}\,.$$

When doing this it is **not** important which variable we choose to use when defining these two functions, h and f. So, whether we write

$$f(t) = 4 - 9t \quad \text{and} \quad h(t) = \sqrt{t}\,,$$

or

$$f(x) = 4 - 9x \quad \text{and} \quad h(x) = \sqrt{x}\,,$$

or even

$$f(x) = 4 - 9x \quad \text{and} \quad h(t) = \sqrt{t}\,,$$

we have still defined the same two functions whose composition $h \circ f$ is R. Of course, just as is always the case, we want to make sure we write things that are meaningful. So, after correctly discovering two functions whose composition is R and giving their definitions as above, we would not want to write something like $R(x) = h(f(t))$. Do you see why this is not really meaningful?

The Chain Rule allows us to compute the derivative of a function like R above and write it in terms of the derivatives of the two "simpler" functions h and f that have the property that $R(x) = (h \circ f)(x)$. The chain rule says that

$$R'(x) = h'(f(x))f'(x)\,,$$

and this should be relatively simple to compute since we already know that $h'(t) = \frac{1}{2}t^{-\frac{1}{2}}$ and $f'(t) = -9$. Putting this together we get

$$R'(x) = h'(f(x))f'(x) = \frac{1}{2}(4-9x)^{-\frac{1}{2}}(-9) = -\frac{9}{2}(4-9x)^{-\frac{1}{2}} = \frac{-9}{2\sqrt{4-9x}}\,.$$

To be successful in using the chain rule then, it is crucial that you be able to accurately see how to decompose a function into the composition of two (or more) "simpler" functions. In each of the following find two functions whose composition is the given one.

1. $F(x) = (x^2 - x + 1)^3$

   If $g(t) = t^2 - t + 1$ and $K(t) = t^3$, then $F(x) = (K \circ g)(x) = K(g(x))$.

2. $P(t) = \sqrt[3]{1 + \tan(t)}$

   Let $L(\theta) = 1 + \tan(\theta)$ and $q(r) = \sqrt[3]{r}$. Then $P(t) = (q \circ L)(t) = q(L(t))$.

3. $n(x) = \sec(5x)$

   If $f(t) = 5t$ and $g(t) = \sec(t)$, then $n(x) = (g \circ f)(x) = g(f(x))$.

4. $g(\theta) = \sin(\frac{1}{\theta})$

   If $M(x) = \sin(x)$ and $N(x) = \frac{1}{x}$, then $g(\theta) = (M \circ N)(\theta) = M(N(\theta))$.

5. $w(\theta) = \frac{1}{\sin(\theta)}$

   Using the same two functions M and N as in 4 above, $w(\theta) = (N \circ M)(\theta) = N(M(\theta))$.

6. $G(x) = \cos(\sqrt{1 + x^2})$

   Let $h(t) = \sqrt{1 + t^2}$ and $f(t) = \cos(t)$, then $G(x) = (f \circ h)(x) = f(h(x))$.

7. $L(t) = \cos(\sin(t))$

   If $f(x) = \sin(x)$ and $g(x) = \cos(x)$, then $L(t) = (g \circ f)(t) = g(f(t))$.

8. $B(x) = (x + x^2)^{12}$

   Let $A(z) = z + z^2$ and $C(z) = z^{12}$. Then $B(x) = (C \circ A)(x) = C(A(x))$.

9. $H(z) = \sin^2(z) + \sin(z)$

   Let $f(x) = x^2 + x$ and let $g(x) = \sin(x)$. Then $H(z) = (f \circ g)(z) = f(g(z))$.