Chapter 1: Fun and Games

Finite Mathematics
Spring 2017

Subtraction Game

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- The players now alternate taking turns removing tokens from the pile.
- The **winner** is the last player to remove a token.

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Play the game and determine a strategy.

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Can you think of a way to determine the winner and a strategy for that player for any given S and n?

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Analyzing Subtraction Games

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- Well, duh??
- The **key idea** is to make up these sets **recursively**.

Example of making $\mathcal{N}$ and $\mathcal{P}$

Example: $\mathbf{S} = \{\mathbf{1}, \mathbf{2}, \mathbf{5}\}$

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- Put $\mathbf{1, 2}$ and $\mathbf{5}$ in $\mathcal{N}$ since the next player can win if confronted with a pile of $\mathbf{1, 2}$ or $\mathbf{5}$ tokens by simply removing all of them.

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- Put $\mathbf{4}$ into $\mathcal{N}$, and then put $\mathbf{6}$ into $\mathcal{P}$. Why?

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Question: Can you do $\mathbf{S} = \{\mathbf{1}, \mathbf{2}, \mathbf{6}, \mathbf{9}\}$?

Summary

1. We build the sets $\mathcal{N}$ and $\mathcal{P}$ recursively.
2. Put 0 and each integer from $\mathbf{S}$ in $\mathcal{N}$.
3. Consider the smallest positive integer $\mathbf{k}$ that has not been put into either $\mathcal{N}$ or $\mathcal{P}$.
4. If there is **at least** one number in $\mathbf{S}$ so that when that many tokens are removed from a pile of $\mathbf{k}$ tokens the number of tokens remaining in the pile is in $\mathcal{P}$, then put $\mathbf{k}$ into $\mathcal{N}$.
5. If the above is not true, then **for every** number in $\mathbf{S}$ when that many tokens are removed from a pile of $\mathbf{k}$ tokens the number of tokens remaining in the pile is in $\mathcal{N}$. Put $\mathbf{k}$ into $\mathcal{P}$.
6. Repeat starting at step 3 above.