SERMON 2006 Problem Session

1. (proposed by Mark Kozek) Suppose $N = d_{n-1}d_{n-2}\cdots d_0$ (written in base 10 with $d_{n-1} \neq 0$) is a positive composite integer coprime to 10 such that if you "insert" any digit $x \in \{0, 1, 2, \ldots, 9\}$ into N (e.g. $d_{n-1}d_{n-2}xd_{n-3}\cdots d_0$ or $xd_{n-1}d_{n-2}\cdots d_0$), the resulting integer is composite.

Does such an N exist if $n \geq 5$? If not, what is the smallest n so that no such N exists?

(addendum by Theresa Vaughn) What if we change the base? And does an N exist where the given condition holds for more than one base simultaneously?

2. (proposed by Robert Rumely) The following is known: Suppose k is a number field, $0 \neq \alpha \in k$ is not a root of unity, S is a finite set of places of k containing the archimedean places. Then there exist only finitely many roots of unity $\xi \in \overline{k}$ which are S-integral with respect to α.

Note that "roots of unity" cannot be replaced by "small points (in height)", because if $k = \mathbb{Q}$, $\alpha = 2$, ξ_n is a root of $f_n(x) = x^{2^n-1}(x-2) - 1$, then $f_n(x+1)$ is 2-Eisenstein, ξ_n is a unit, and $\xi_n - 2 = 1/\xi_n^{2^n-1}$ is a unit ($h(\xi_n) \approx C/2^n = C/\deg(f)$).

Would a sequence of points violating Lehmer's conjecture have the finiteness property?

3. (proposed by Bruce Berndt) Let N be a positive integer, $S = \{\text{orange integers}\} \cup \{\text{blue multiples of } 7\}$, $A(N)$ the number of partitions of $2N$ into distinct even parts of S and $B(N)$ the number of partitions of $2N+1$ into distinct odd parts of S. Farkas and Kra first proved that $A(N) = B(N)$ (Warnaar and Hirschhorn have also given proofs); give a combinatorial proof of this fact.

4. (proposed by Griff Elder) The following is known: Suppose $L/K/\mathbb{Q}_p$ are finite Galois extensions, with L/K elementary abelian. Letting

$$G_i = \{\sigma \in Gal(L/K) \mid (\sigma - 1)\pi_L \in P_L^{i-1}\},$$

suppose there is only one break in the ramification filtration, i.e.

$$G = G_{-1} = G_0 = \cdots = G_b \supset\neq G_{b+1} = \cdots$$

There exists $\alpha \in L$ such that $L = K[G]_\alpha$. If $v_L(\alpha) = b$, then α is a normal field basis generator.

Does the same result hold in the function field case (i.e. $L/K/\mathbb{F}_q((t))$)?

5. (proposed by Gang Yu) For $n \in \mathbb{N}$, suppose $f(x) = \sum_{j=0}^{n} a_j x^j$ with $a_j \in \{0, 1\}$ and $a_n = 1$. Write $f^2(x) = \sum_{j=0}^{2n} b_j x^j$. Prove or disprove:

$$\max_{0 \le j \le 2n} b_j >_\sim f^2(1)/n$$

6. (proposed by David Penniston) Let $b_p(n)$ for p prime denote the number of p-regular partitions of n (i.e. the number of partitions of n none of whose parts is divisible by p). I can show that for at least $4/5$ (resp. $10/11$) of the values of n, $b_{11}(n)$ (resp. $b_{23}(n)$ is divisible by 5 (resp. 11). Give a combinatorial explanation for why we have these frequent divisibilities.