It’s a Small World After All
Calculus without ϵ’s and δ’s

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L'Hôpital's axiom

- Guillaume François Antoine Marquis de l'Hôpital (1661 - 1704) wrote the first calculus textbook, *Analyse des infiniment petits pour l'intelligence des lignes courbes*, in 1696.
- First axiom:
  - Demande ou supposition: On demande qu'on puisse prendre indifféremment l'une pour l'autre deux quantités qui ne différent entr'elles que d'une quantité infiniment petite ...
- That is: if a and b are real numbers and $a - b$ is *infinitely small*, then we may take $a = b$.

Newton (1642 - 1727)

- To find the derivative of $y = x^2$, Newton begins by computing

$$(x+o)^2 - x^2 = (x^2 + 2xo + o^2) - x^2 = 2xo + o^2.$$

where o is assumed to be a very small increment in x.

- He then discards the term o^2, essentially saying it "vanishes" because it is a power of a very small number.
- Dividing by o, he now has

$$\frac{(x+o)^2 - x^2}{o} = 2x,$$

which he takes to be the desired derivative.

Newton’s explanation

- In Newton’s words:

  First those termes ever vanish which are not multiplied by o, they being the propounded equation. Secondly those termes also vanish in which o is of more than one dimension, because they are infinitely lesse than those in which o is but of one dimension. Thirdly the still remaining termes, being divided by o will have [the desired form].

Another explanation

- In other places, Newton comes close to stating the modern notion of a limit:

  > By the ultimate ratio of evanescent quantities (i.e., ones that are approaching zero) is to be understood the ratio of the quantities not before they vanish, nor afterwards, but with which they vanish . . . . Those ultimate ratios with which quantities vanish are not truly the ratios of ultimate quantities, but limits towards which the ratios of quantities decreasing without limit do always converge; and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished *ad infinitum*.

Leibniz (1646 - 1716)

- Leibniz’s take:

  Whether infinite extensions successively greater and greater, or infinitely small ones successively less and less, are legitimate considerations, is a matter that I own to be possibly open to question; but for him who would discuss these matters, it is not necessary to fall back upon metaphysical controversies, such as the composition of the continuum, or to make general geometrical matters depend thereon.

Leibniz (cont'd)

- More:

  It will be sufficient if, when we speak of infinitely great (or more strictly unlimited), or of infinitely small quantities (i.e., the very least of those within our knowledge), it is understood that we mean quantities that are indefinitely great or indefinitely small, i.e., as great as you please, or as small as you please, so that the error that any one may assign may be less than a certain assigned quantity.

Leibniz (cont'd)

- More:

  If any one wishes to understand these [the infinitely great and the infinitely small] as the ultimate things, or as truly infinite, it can be done, and that too without falling back upon a controversy about the reality of extensions, or of infinite continuums in general, or of the infinitely small, ay, even though he think that such things are impossible; it will be sufficient simply to make use of them as a tool that has advantages for the purpose of the calculation, just as the algebraists retain imaginary roots with great profit. For they contain a handy means of reckoning, as can manifestly be verified in every case in a rigorous manner by the method already stated.

Weierstraß (1815 - 1897)

- In the 1860’s, Karl Weierstraß showed how to develop calculus without direct reference to either infinitely large or infinitely small numbers.
- His ϵ-δ formulation of caclulus became the norm for mathematical analysis.

Robinson (1918 - 1974)

- In 1961, Abraham Robinson showed how to develop calculus in a logically conistent manner starting with a continuum containing both infinitely small and infinitely large numbers.
- This new approach to analysis is now called *non-standard analysis*.

Rational numbers

- Let $\mathbb{N}$ be the set of positive integers.
- Let $\mathbb{Z}$ be the set of all integers.
- Leopold Kronecker (1821 - 1891): God made the integers; all else is the work of man.
- We define $\mathbb{Q}$, the set of *rational numbers*, as follows:
  - Let S be the set $\{(p,q) : p,q \in \mathbb{Z}, q \neq 0\}$.
  - Define an equivalence relation on S: $(p,q) \sim (r,s)$ if $ps = qr$.
  - $\mathbb{Q}$ is the set of all equivalence classes of S.

Real numbers: Dedekind cuts

- Richard Dedekind (1831 - 1916) first defined real numbers in terms of partitions of the rational numbers:
  - We call disjoint sets $C_1, C_2 \subset \mathbb{Q}$ a *Dedekind cut* if $C_1 \cup C_2 = \mathbb{Q}$, C_1 has no greatest element, and for every $x \in C_1$ and $y \in C_2$, $x < y$.
  - The set of *real numbers*, $\mathbb{R}$, is the set of all Dedekind cuts.
- Example: $\sqrt{2}$ is the Dedekind cut consisting of $C_1 = \{r \in \mathbb{Q} : r < 0$ or $r^2 < 2\}$ and $C_2 = \mathbb{Q} - C_1$.

Real numbers: Cauchy sequences

- We say a sequence $\{r_n\}$ in $\mathbb{Q}$ is a *Cauchy sequence* if for every rational number $\epsilon > 0$ there exists $N \in \mathbb{Z}$ such that $|r_n - r_m| < \epsilon$ whenever $n, m > N$.
- Let S be the set of all Cauchy sequences in $\mathbb{Q}$.
- Define an equivalence relation on S: $\{r_n\} \sim \{s_n\}$ if given any $\epsilon > 0$ there exists $N \in \mathbb{Z}$ such that $|r_n - s_n| < \epsilon$ for all $n > N$.
- $\mathbb{R}$ is the set of all equivalence classes of S.
- Example: $\sqrt{2}$ is the equivalance class of

$$\{1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414212, \ldots\}.$$

Zero

- Note: We identify $r \in \mathbb{Q}$ with $\{r, r, r, r, \ldots\}$.
- Example:

$$\left\{\frac{1}{n}\right\} \sim \left\{\frac{1}{n^2}\right\} \sim \{0, 0, 0, \ldots\} = 0.$$

- The equivalance relation is not sensitive enough to distinquish different rates of convergence to 0.

Filters

- If I is a nonempty set, then the *power set* of I is

$$\mathcal{P}(I) = \{A : A \subset I\}.$$

- We call $\mathcal{F} \subset \mathcal{P}(I)$ a *filter* if
  - $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$
  - $A \in \mathcal{F}$ and $A \subset B \Rightarrow B \in \mathcal{F}$.
- We call a filter $\mathcal{F}$ a *proper filter* if $\emptyset \notin \mathcal{F}$.
- We call a proper filter an *ultrafilter* if for any $A \subset I$, either $A \in \mathcal{F}$ or $I - A \in \mathcal{F}$.

Examples

- For any $i \in I$,

$$\mathcal{F}^i = \{A \subset I : i \in A\}$$

  is an ultrafilter, called the *principal ultrafilter generated by* i.

- The set

$$\mathcal{F}^{\mathrm{co}} = \{A \subset I : I - A \text{ is finite}\}$$

  is a filter, called the *cofinite*, or *Fréchet*, filter.

- Note: $\mathcal{F}^{\mathrm{co}}$ is proper if and only if I is infinite.
- Note: $\mathcal{F}^{\mathrm{co}}$ is not an ultrafilter.

Nonprincipal ultrafilters

- Theorem: If I is an infinite set, then there exists a nonprincipal ultrafilter on I.
- Proof: Apply Zorn's lemma to the collection of all proper filters which contain $\mathcal{F}^{\mathrm{co}}$.
- Note: If $\mathcal{F}$ is a nonprincipal ultrafilter and $A \in \mathcal{F}$, then A is infinite.
- From now on, we let $\mathcal{F}$ be a fixed nonprincipal ultrafilter on $\mathbb{N}$.

Hyperreal numbers

- Let S be the set of all sequences of real numbers.
- Define an equivalence relation in S: $\{a_n\} \sim \{b_n\}$ if

$$\{n \in \mathbb{N} : a_n = b_n\} \in \mathcal{F}.$$

- The *hyperreal numbers*, ${}^*\mathbb{R}$, is the set of all equivalence classes of S.
- Note: we identify $a \in \mathbb{R}$ with the equivalence class of the sequence $\{a, a, a, \ldots\}$.

Algebraic operations

- Notation: Let $[r]$ be the equivalence class of the sequence r.
- If $r = \{r_1, r_2, r_3, \ldots\}$ and $s = \{s_1, s_2, s_3, \ldots\}$, define

$$[r] + [s] = [\{r_1 + s_1, r_2 + s_2, r_3 + s_3, \ldots\}]$$

  and

$$[r] \times [s] = [\{r_1 s_1, r_2 s_2, r_3 s_3, \ldots\}].$$

- Note: if $r = \{1, 0, 1, 0, 1, \ldots\}$ and $s = \{0, 1, 0, 1, 0, \ldots\}]$, then

$$[r] \times [s] = [\{0, 0, 0, 0, 0, \ldots\}] = 0.$$

- Note: either $\{2, 4, 6, 8, \ldots\} \in \mathcal{F}$ or $\{1, 3, 5, 7, \ldots\} \in \mathcal{F}$, but not both. Hence either $[r] = 0$ and $[s] = 1$, or $[r] = 0$ and $[s] = 1$.

Order

- We write $[r] < [s]$ if

$$\{i \in \mathbb{N} : r_i < s_i\} \in \mathcal{F}.$$

- Example: if $\epsilon = \{\frac{1}{n}\}$ and $r \in \mathbb{R}$, $r > 0$, then $0 < [\epsilon] < r$.
- Example: if $\gamma = \{\frac{1}{n^2}\}$, then $0 < \gamma < \epsilon$
- Example: if $\omega = \{n\}$ and $r \in \mathbb{R}$, then $[\omega] > r$.
- Note: $[\epsilon] \times [\omega] = 1$, so

$$[\omega] = \frac{1}{[\epsilon]} \text{ and } [\epsilon] = \frac{1}{[\omega]}.$$

Definitions

- We call a hyperreal number ϵ with $|\epsilon| < r$ for every positive real number r an *infinitesimal*.
- Note: 0 is the only infinitesimal real number.
- We call a hyperreal number ω with $|\omega| > r$ for every real number r an *unlimited* hyperreal number. A hyperreal number which is not unlimited is *limited*

Enlarging sets and functions

- If $A \subset \mathbb{R}$, define ${}^*A$ by

$$[r] \in {}^*A \text{ if and only if } \{n \in \mathbb{N} : r_n \in A\} \in \mathcal{F}.$$

- If $f : A \to \mathbb{R}$, define ${}^*f : {}^*A \to {}^*\mathbb{R}$ by

$${}^*f([r]) = [f(r_n)].$$

- Theorem: A is finite if and only if $A = {}^*A$.

Example

- Both $\omega = [\{n\}]$ and $w^2 = [\{n^2\}]$ are in ${}^*\mathbb{N}$.
- Note: if $m \in \mathbb{N}$, $\omega + m < \omega^2$.

Properties of the hyperreals

- ${}^*\mathbb{R}$ is an ordered field.
- Definition: an ordered field F is *Archimedean* if for every $x, y \in F$ with $0 < x < y$, there exists an $n \in \mathbb{N}$ such that $nx > y$.
- ${}^*\mathbb{R}$ is Archimedean if we allow $n \in {}^*\mathbb{N}$.
- ${}^*\mathbb{R}$ does not have the least upper bound property.
  - The set $\{x \in \mathbb{R} : x < 0\}$ does not have a least upper bound in ${}^*\mathbb{R}$.
- If ϵ is infinitesimal and b is limited, then ϵb is infinitesimal.

Some terminology

- We write $b \simeq c$ to mean $b - c$ is infinitesimal.
- For any $b \in {}^*\mathbb{R}$, we call

$$\mathrm{hal}(b) = \{c \in {}^*\mathbb{R} : c \simeq b\}$$

  the *halo* of b.
- Robinson called $\mathrm{hal}(b)$ the *monad* of b in honor of Leibniz.
- Theorem: If b is a limited hyperreal number, then there exists a unique real number r for which $r \in \mathrm{hal}(b)$.
- We call the r in the theorem the *shadow* of b, denoted $\mathrm{sh}(b)$.
- Some refer to $\mathrm{sh}(b)$ as the *standard part* of b.

Continuity

- We say a function f is continuous at $c \in \mathbb{R}$ if, for any infinitesimal ϵ, $f(c+\epsilon) \simeq f(c)$.
- That is: $f(\mathrm{hal}(c)) \subset \mathrm{hal}(f(c))$.
- Example: If $f(x) = x^2$, then

$$f(c+\epsilon) = (c+\epsilon)^2 = c^2 + 2\epsilon c + \epsilon^2 \simeq c^2,$$

so f is continuous at any real number c

Example

- Let $f(x) = \frac{1}{x}$.
- If ϵ is infinitesimal,

$$f(c+\epsilon) = \frac{1}{c+\epsilon} = \frac{1}{c} - \frac{\epsilon}{c(c+\epsilon)}.$$

- If $c \in \mathbb{R}$, $c \neq 0$, $f(c+\epsilon) \simeq f(c)$, and f is continuous at c.
- However, if, for example, $c = \epsilon$,

$$f(\epsilon+\epsilon) - f(\epsilon) = -\frac{1}{2\epsilon},$$

  which is unlimited.
- Reason: f is continuous, but not *uniformly continuous*.

Derivatives

- Given a function $y = f(x)$ and a nonzero infinitesimal dx, let
$$dy = f(x + dx) - f(x).$$
- If $\mathrm{sh}\left(\frac{dy}{dx}\right)$ is the same for all infinitesimals dx, we call
$$f'(x) = \mathrm{sh}\left(\frac{dy}{dx}\right) = \mathrm{sh}\left(\frac{f(x+dx) - f(x)}{dx}\right)$$
  the *derivative* of f at x.

Example

- If $y = x^2$, then

$$\begin{aligned} dy &= (x + dx)^2 - x^2 \\ &= x^2 + 2xdx + dx^2 - x^2 \\ &= 2xdx + dx^2, \end{aligned}$$

  and so $\dfrac{dy}{dx} = 2x + dx \simeq 2x$.
- Note:

$$d^2y = d(dy) = (2(x + dx)dx + dx^2) - (2xdx + dx^2) = 2dx^2$$

- And so: $\dfrac{d^2y}{dx^2} = 2$.

Product rule

- Suppose u and v are differentiable.
- Then

$$d(uv) = (u+du)(v+dv) - uv = udv + vdu + dudv.$$

- Hence

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} + du\frac{du}{dx} \simeq u\frac{dv}{du} + v\frac{du}{dx},$$

since du is infinitesimal and $\frac{du}{dx}$ is limited.

Quotient rule

- If $y = \frac{u}{v}$, then $u = vy$, so

$$du = vdy + ydv.$$

- Hence

$$dy = \frac{du - ydv}{v} = \frac{du - \frac{u}{v}dv}{v} = \frac{vdu - udv}{v^2}.$$

- Thus

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Definite integral

- Suppose f is continuous on $[a, b]$.
- Let $N \in {}^*\mathbb{N}$ be unlimited, and let

  $$a = x_0 < x_1 < x_2 < \cdots < x_N = b$$

  be a partition of $[a, b]$ into N subintervals. Let x_i^* be a point in the ith subinterval.
- Then we may define

  $$\int_a^b f(x)dx = \sum_{i=1}^{N} f(x_i^*)(x_i - x_{i-1}).$$

Cauchy (1789 - 1857)

- From Cauchy's *Cours d'Analyse*:
  - Lorsque les valeurs numériques successives d'une même variable décroissent indf́iniment, de maniéire à s'abaisser au'dessous de tout nombre donné, cette variable devient ce qu'on nomme un *infiniment petit* ou une quantité *infiniment petite*.
  - ... la fonction $f(x)$ restera continue par rappport à x entre les limites données, si, entre ces limites, un accroissement infiniment petit de la variable produit toujuors un accroissement infiniment petit de la fonction ell-même.

Cours d’analyse (cont’d)

- Lorsque les différents termes de la série

$$u_0, u_1, u_2, \ldots, u_n, u_{n+1}, \ldots$$

sont des fonctions d’une même variable x, continues par rapport à cette variable, dans le voisinage d’une valeur particuliére pour laquelle la série est convergent, la somme de la série est aussi, dans le voisinage de cette valeur particulière, fonction continue de x.

Counterexample

- The function

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$$

has jump discontinuites at $x = 0, \pm 2\pi, \pm 4\pi, \ldots$.

Cauchy's reply (1853)

- Si les différents termes de la série

$$u_0, u_1, u_2, \ldots, u_n, u_{n+1}, \ldots$$

sont des fonctions de la variable réelle x, continues, par rapport à cette variable, entre des limited données; si, d'ailleurs, la somme

$$u_n + u_{n+1} + \cdots + u_{n'+1}$$

devient toujours infiniment petite pour des valeurs infiniment grandes des nombres entiers n et $n' > n$, la série sera convergente, et la somme s de la série sera, entere les limites données, fonction continu de la variable x.

Counterexample revisited

- Let

$$r_n(x) = \sum_{k=n}^{\infty} \frac{\sin(kx)}{k}.$$

- If $x = \frac{1}{n}$, then

$$\begin{aligned} r_n(x) &= \sum_{k=n}^{\infty} \frac{\sin\left(\frac{k}{n}\right)}{k} \\ &= \sum_{k=n}^{\infty} \frac{\sin\left(\frac{k}{n}\right)}{\frac{k}{n}} \frac{1}{n} \\ &\xrightarrow[n\to\infty]{} \int_1^{\infty} \frac{\sin(x)}{x} dx \approx 0.624713. \end{aligned}$$

Counterexample (cont'd)

- Hence, for Cauchy, the series does not converge at $\{\frac{1}{n}\}$.
- That is, the series does not coverge for all points in a neighborhood of 0 if you allow for infinitesimals.

Euclid's theorem

- A non-standard proof that there are an infinite number of prime numbers:
  - Let $\Pi \subset \mathbb{R}$ be the set of all prime numbers.
  - Let $N = (1)(2)(3)(4)\cdots + 1$.
  - For every $p \in \Pi$, p does not divide N.
  - Hence there exists $p \in {}^*\Pi$ for which $p \notin \Pi$.
  - Thus Π must be infinite.