The *De Continuo* of Thomas Bradwardine

Dan Sloughter

Furman University

7 January 2008

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In this matter, and great disputation,
And has been (disputed) by a hundred thousand men.
But I can not separate the valid and invalid arguments
As can the holy doctor Augustine,
Or Boethius, or the Bishop Bradwardyn,
Whether God's worthy foreknowledge
Constrains me by need to do a thing –
"Need" I call simple necessity –
Or else, if free choice be granted to me
To do that same thing, or do it not,
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  - Implicitly in a discussion of the impossibility of dividing an indivisible in the *Summoner's Tale*.

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- Wrote a well-known book on proportions, *Tractatus de proportionibus*.

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- A continuum may be geometrical (lines, surfaces, bodies), physical (space), or temporal (time, or motion).

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- For example, is a line just a union of points?

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- The last of these is the standard modern answer, at least since the latter part of the 19th century.

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- Hence all continua are composed in the same way as time.

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  - And from the *Century Dictionary* of 1889:
    - [Continuous means] in mathematics and philosophy a connection of points (or other elements) as intimate as that of the instants or points of an interval of time: thus, the continuity of space consists in this, that a point can move from any one position to any other so that at each instant it shall have a definite and distinct position in space.

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- It follows that, if line segments are composed from indivisibles, then two line segments composed from an equal number of indivisibles must have the same length.

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  - Then Bradwardine's conclusion is that

$$\text{Area of } A \geq \text{Area of } B.$$

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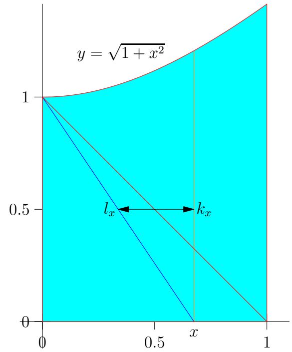
  - Let $B = \bigcup_{x\in[0,1]} k_x$.
  - Now

$$T = \bigcup_{x\in[0,1]} \ell_x \text{ and } m(\ell_x) = m(k_x)$$

for all $x \in [0,1]$.

An example (cont'd)



- Hence, by Bradwardine's 133rd Conclusion,

$$\text{Area of } T = \text{Area of } B > \text{Area of } S = 2(\text{Area of } T).$$

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  - Grünbaum argued that the essence of Zeno's mistake is that measures are countably additive, but not uncountably additive.
  - But why are measures restricted to, at most, countable additivity?

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- That is, lines are composed from lines and planar regions from planar regions.

Bradwardine's option

- Corollary to Conclusion 141: A continuum is composed, not from indivisible parts, but from parts of the same type.
  - Omne continuum ex infinitis continuis similis speciei cum illo componi.
  - Every continuum is composed from an infinite number of continua of similar type with itself.
- That is, lines are composed from lines and planar regions from planar regions.
- An infinite number? What type of infinity?

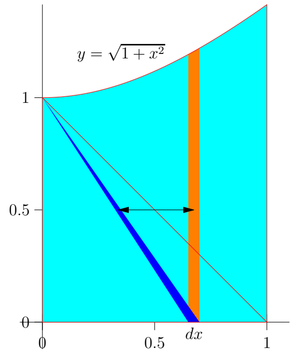
Bradwardine's option (cont'd)

- Consider the one-to-one correspondence $\ell_x \leftrightarrow k_x$ as a transformation between two-dimension regions instead of as a transformation between lines.

Bradwardine's option (cont'd)

- Consider the one-to-one correspondence $\ell_x \leftrightarrow k_x$ as a transformation between two-dimension regions instead of as a transformation between lines.
- That is, starting with an infinitesimal part dx of the segment $[0, 1]$, consider the mapping which takes the triangle τ_{dx} with base dx and upper vertex at $(0, 1)$ to the region ρ_{dx} with base dx which extends up to the curve $y = \sqrt{1 + x^2}$.

Bradwardine's option (cont'd)



- Then T is composed from the infinitesimal triangles τ_{dx} and B is composed from the infinitesimal regions ρ_{dx}.

Bradwardine's option (cont'd)

- Moreover, since τ_{dx} has area $\frac{1}{2}dx$,

$$\text{Area of } T = \int_0^1 \frac{1}{2}dx = \frac{1}{2}.$$

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- There is no contradiction because τ_{dx} and ρ_{dx} do not have the same area.

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Peirce, Kant, continuity

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  - Kant's real definition implies that a continuous line contains no points. Now if we are to accept the common sense idea of continuity (after correcting its vagueness and fixing it to mean something) we must either say that a continuous line contains no points or we must say that the principle of excluded middle does not hold of these points.

Peirce, Kant, continuity (cont'd)

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- And:
  - In the calculus and theory of functions it is assumed that between any two rational points (or points at distances along the line expressed by rational fractions) there are rational points and that further for every convergent series of such fractions (such as 3.1, 3.14, 3.141, 3.1415, 3.14159, etc.) there is just one limiting point; and such a collection of points is called continuous. But this does not seem to be the common sense idea of continuity. It is only a collection of independent points. Breaking grains of sand more and more will only make the sand more broken. It will not weld the grains into unbroken continuity.