The Calculus of Functions

of

Several Variables

Section 4.4

Green's Theorem

Green's theorem is an example from a family of theorems which connect line integrals (and their higher-dimensional analogues) with the definite integrals we studied in Section 3.6. We will first look at Green's theorem for rectangles, and then generalize to more complex curves and regions in $\mathbb{R}^2$.

Green's theorem for rectangles

Suppose $F : \mathbb{R}^2 \to \mathbb{R}^2$ is C^1 on an open set containing the closed rectangle

$$D = [a, b] \times [c, d],$$

and let F_1 and F_2 be the coordinate functions of F. If C denotes the boundary of D, oriented in the clockwise direction, then we may decompose C into the four curves C_1, C_2, C_3, and C_4 shown in Figure 4.4.1. Then

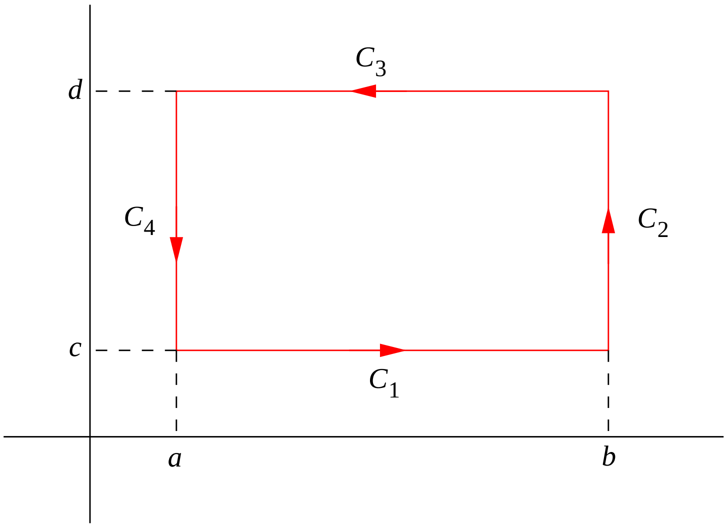

Figure 4.4.1 The boundary of a rectangle decomposed into four smooth curves

$$\alpha(t) = (t, c),$$

$a \leq t \leq b$, is a smooth parametrization of C_1,

$$\beta(t) = (b, t),$$

$c \le t \le d$, is a smooth parametrization of C_2,

$$\gamma(t) = (t, d),$$

$a \le t \le b$, is a smooth parametrization of $-C_3$, and

$$\delta(t) = (a, t),$$

$c \le t \le d$, is a smooth parametrization of $-C_4$. Now

$$\begin{aligned}\int_C F\cdot ds &= \int_{C_1} F\cdot ds + \int_{C_2} F\cdot ds + \int_{C_3} F\cdot ds + \int_{C_4} F\cdot ds \\ &= \int_{C_1} F\cdot ds + \int_{C_2} F\cdot ds - \int_{-C_3} F\cdot ds - \int_{-C_4} F\cdot ds, \end{aligned} \tag{4.4.1}$$

and

$$\int_{C_1} F\cdot ds = \int_a^b ((F_1(t,c), F_2(t,c))\cdot(1,0)dt = \int_a^b F_1(t,c)dt, \tag{4.4.2}$$

$$\int_{C_2} F\cdot ds = \int_c^d ((F_1(b,t), F_2(b,t))\cdot(0,1)dt = \int_c^c F_2(b,t)dt, \tag{4.4.3}$$

$$\int_{-C_3} F\cdot ds = \int_a^b ((F_1(t,d), F_2(t,d))\cdot(1,0)dt = \int_a^b F_1(t,d)dt, \tag{4.4.4}$$

and

$$\int_{-C_4} F\cdot ds = \int_c^d ((F_1(a,t), F_2(a,t))\cdot(0,1)dt = \int_c^c F_2(a,t)dt, \tag{4.4.5}$$

Hence, inserting (4.4.2) through (4.4.5) into (4.4.1),

$$\begin{aligned}\int_C F\cdot ds &= \int_a^b F_1(t,c)dt + \int_c^d F_2(b,t)dt - \int_a^b F_1(t,d)dt - \int_c^d F_2(a,t)dt \\ &= \int_c^d (F_2(b,t) - F_2(a,t))dt - \int_a^b (F_1(t,d) - F_1(t,c))dt. \end{aligned} \tag{4.4.6}$$

Now, by the Fundamental Theorem of Calculus, for a fixed value of t,

$$\int_a^b \frac{\partial}{\partial x} F_2(x,t)dx = F_2(b,t) - F_2(a,t) \tag{4.4.7}$$

and

$$\int_c^d \frac{\partial}{\partial y} F_1(t,y)dy = F_1(t,d) - F_1(t,c). \tag{4.4.8}$$

Thus, combining (4.4.7) and (4.4.8) with (4.4.6), we have

$$\begin{aligned}\int_C F \cdot ds &= \int_c^d \int_a^b \frac{\partial}{\partial x} F_2(x,t)dxdt - \int_a^b \int_c^d \frac{\partial}{\partial y} F_1(t,y)dydt \\ &= \int_c^d \int_a^b \frac{\partial}{\partial x} F_2(x,y)dxdy - \int_a^b \int_c^d \frac{\partial}{\partial y} F_1(x,y)dydx \\ &= \int_c^d \int_a^b \left(\frac{\partial}{\partial x} F_2(x,y) - \frac{\partial}{\partial y} F_1(x,y) \right) dxdy. \qquad (4.4.9)\end{aligned}$$

If we let $p = F_1(x,y)$, $q = F_2(x,y)$, and $\partial D = C$ (a common notation for the boundary of D), then we may rewrite (4.4.9) as

$$\int_{\partial D} pdx + qdy = \int\int_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) dxdy. \qquad (4.4.10)$$

This is Green's theorem for a rectangle.

Example If $D = [1,3] \times [2,5]$, then

$$\begin{aligned}\int_{\partial D} xydx + xdy &= \int\int_D \left(\frac{\partial}{\partial x} x - \frac{\partial}{\partial y} xy \right) dxdy \\ &= \int_1^3 \int_2^5 (1-x)dydx \\ &= \int_1^3 3(1-x)dx \\ &= 3x\Big|_1^3 - \frac{3}{2}x^2\Big|_1^3 \\ &= -6.\end{aligned}$$

Clearly, this is simpler than evaluating the line integral directly.

Green's theorem for regions of Type III

Green's theorem holds for more general regions than rectangles. We will confine ourselves here to discussing regions known as regions of Type III, but it is not hard to generalize to regions which may be subdivided into regions of this type (for an example, see Problem 12). Recall from Section 3.6 that we say a region D in $\mathbb{R}^2$ is of Type I if there exist real numbers $a < b$ and continuous functions $\alpha : \mathbb{R} \to \mathbb{R}$ and $\beta : \mathbb{R} \to \mathbb{R}$ such that

$$D = \{(x,y) : a \leq x \leq b, \alpha(x) \leq y \leq \beta(x)\}. \qquad (4.4.11)$$

We say a region D in $\mathbb{R}^2$ is of Type II if there exist real numbers c and d and continuous functions $\gamma : \mathbb{R} \to \mathbb{R}$ and $\delta : \mathbb{R} \to \mathbb{R}$ such that

$$D = \{(x,y) : c \leq y \leq d, \gamma(y) \leq x \leq \delta(y)\}. \qquad (4.4.12)$$

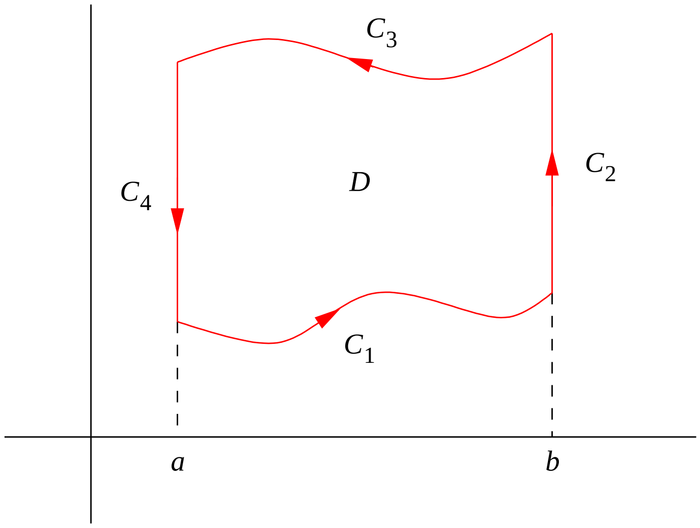

Figure 4.4.2 Decomposing the boundary of a region of Type I

Definition We call a region D in $\mathbb{R}^2$ which is both of Type I and of Type II a region of *Type III.*

Example In Section 3.6, we saw that the triangle T with vertices at $(0,0)$, $(1,0)$, and $(1,1)$ and the closed disk

$$D = \bar{B}^2((0,0),1) = \{(x,y) : x^2 + y^2 \leq 1\}$$

are of both Type I and Type II. Thus T and D are regions of Type III. We also saw that the region E beneath the graph of $y = x^2$ and above the interval $[-1,1]$ is of Type I, but not of Type II. Hence E is not of Type III.

Example Any closed rectangle in $\mathbb{R}^2$ is a region of Type III, as is any closed region bounded by an ellipse.

Now suppose D is a region of Type III and ∂D is the boundary of D, that is, the curve enclosing D, oriented counterclockwise. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 vector field, with coordinate functions $p = F_1(x,y)$ and $q = F_2(x,y)$. We will first prove that

$$\int_{\partial D} p dx = -\int\int_D \frac{\partial p}{\partial y} dx dy. \tag{4.4.13}$$

Since D is, in particular, a region of Type I, there exist continuous functions α and β such that

$$D = \{(x,y) : a \leq x \leq b, \alpha(x) \leq y \leq \beta(x)\}. \tag{4.4.14}$$

In addition, we will assume that α and β are both differentiable (without this assumption the line integral of F along ∂D would not be defined). As with the rectangle in the previous proof, we may decompose ∂D into four curves, C_1, C_2, C_3, and C_4, as shown in Figure 4.4.2. Then

$$\varphi_1(t) = (t, \alpha(t)),$$

$a \le t \le b$, is a smooth parametrization of C_1,

$$\varphi_2(t) = (b, t),$$

$\alpha(b) \le t \le \beta(b)$, is a smooth parametrization of C_2,

$$\varphi_3(t) = (t, \beta(t)),$$

$a \le t \le b$, is a smooth parametrization of $-C_3$, and

$$\varphi_4(t) = (a, t),$$

$\alpha(a) \le t \le \beta(a)$, is a smooth parametrization of $-C_4$. Now

$$\int_{\partial D} p\,dx = \int_{C_1} p\,dx + \int_{C_2} p\,dx - \int_{-C_3} p\,dx - \int_{-C_4} p\,dx, \tag{4.4.15}$$

where

$$\int_{C_1} p\,dx = \int_a^b (F_1(t, \alpha(t)), 0) \cdot (1, \alpha'(t))dt = \int_a^b F_1(t, \alpha(t))dt, \tag{4.4.16}$$

$$\int_{C_2} p\,dx = \int_{\alpha(b)}^{\beta(b)} (F_1(b, t), 0) \cdot (0, 1)dt = \int_{\alpha(b)}^{\beta(b)} 0dt = 0, \tag{4.4.17}$$

$$\int_{-C_3} p\,dx = \int_a^b (F_1(t, \beta(t)), 0) \cdot (1, \beta'(t))dt = \int_a^b F_1(t, \beta(t))dt, \tag{4.4.18}$$

and

$$\int_{-C_4} p\,dx = \int_{\alpha(a)}^{\beta(a)} (F_1(a, t), 0) \cdot (0, 1)dt = \int_{\alpha(a)}^{\beta(a)} 0dt = 0. \tag{4.4.19}$$

Hence

$$\begin{aligned} \int_{\partial D} p\,dx &= \int_a^b F_1(t, \alpha(t))dt - \int_a^b F_1(t, \beta(t))dt \\ &= -\int_a^b (F_1(t, \beta(t)) - F_1(t, \alpha(t)))dt. \end{aligned} \tag{4.4.20}$$

Now, by the Fundamental Theorem of Calculus,

$$\int_{\alpha(t)}^{\beta(t)} \frac{\partial}{\partial y} F_1(t, y)dy = F_1(t, \beta(t)) - F_1(t, \alpha(t)), \tag{4.4.21}$$

and so

$$\begin{aligned} \int_{\partial D} p\,dx &= -\int_a^b \int_{\alpha(t)}^{\beta(t)} \frac{\partial}{\partial y} F_1(t, y)dydt \\ &= -\int_a^b \int_{\alpha(x)}^{\beta(x)} \frac{\partial}{\partial y} F_1(x, y)dydx \\ &= -\int\int_D \frac{\partial p}{\partial y}\,dxdy. \end{aligned} \tag{4.4.22}$$

A similar calculation, treating D as a region of Type II, shows that

$$\int_{\partial D} q dy = \int\int_D \frac{\partial q}{\partial x}\, dxdy. \tag{4.4.23}$$

(You are asked to verify this in Problem 7.) Putting (4.4.22) and (4.4.23) together, we have

$$\begin{aligned}\int_{\partial D} F \cdot ds = \int_{\partial D} pdx + qdy &= -\int\int_D \frac{\partial p}{\partial y}\, dxdy + \int\int_D \frac{\partial q}{\partial x}\, dxdy \\ &= \int\int_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dxdy. \end{aligned} \tag{4.4.24}$$

Green's Theorem Suppose D is a region of Type III, ∂D is the boundary of D with counterclockwise orientation, and the curves describing ∂D are differentiable. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 vector field, with coordinate functions $p = F_1(x, y)$ and $q = F_2(x, y)$. Then

$$\int_{\partial D} pdx + qdy = \int\int_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dxdy. \tag{4.4.25}$$

Example Let D be the region bounded by the triangle with vertices at $(0, 0)$, $(2, 0)$, and $(0, 3)$, as shown in Figure 4.4.3. If we orient ∂D in the counterclockwise direction, then

$$\begin{aligned}\int_{\partial D} (3x^2 + y)dx + 5xdy &= \int\int_D \left(\frac{\partial}{\partial x}(5x) - \frac{\partial}{\partial y}(3x^2 + y)\right) dxdy \\ &= \int\int_D (5 - 1)dxdy \\ &= 4\int\int_D dxdy \\ &= (4)(3) \\ &= 12, \end{aligned}$$

where we have used the fact that the area of D is 3 to evaluate the double integral.

The line integral in the previous example reduced to finding the area of the region D. This can be exploited in the reverse direction to compute the area of a region. For example, given a region D with area A and boundary ∂D, it follows from Green's theorem that

$$A = \int\int_D dxdy = \int_{\partial D} pdx + qdy \tag{4.4.26}$$

for any choice of p and q which have the property that

$$\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} = 1. \tag{4.4.27}$$

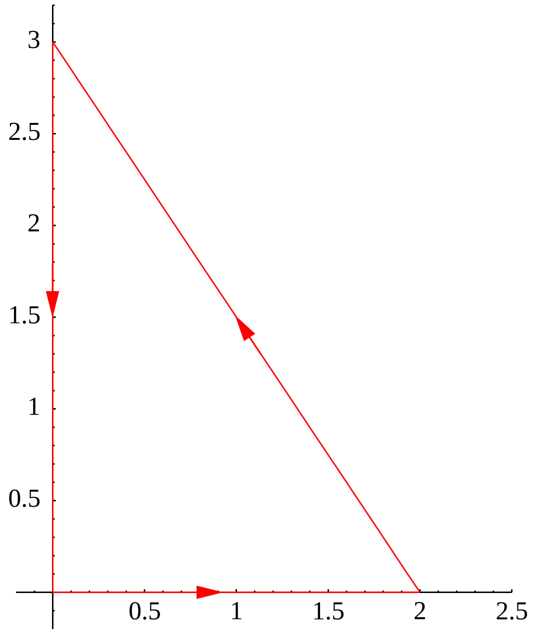

Figure 4.4.3 A triangle with counterclockwise orientation

For example, letting $p = 0$ and $q = x$, we have

$$A = \int_{\partial D} x dy \tag{4.4.28}$$

and, letting $p = -y$ and $q = 0$, we have

$$A = -\int_{\partial D} y dx. \tag{4.4.29}$$

The next example illustrates using the average of (4.4.28) and (4.4.29) to find A:

$$A = \frac{1}{2}\left(\int_{\partial D} x dy - \int_{\partial D} y dx\right) = \frac{1}{2}\int_{\partial D} x dy - y dx. \tag{4.4.30}$$

Example Let A be the area of the region D bounded by the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a > 0$ and $b > 0$, as shown in Figure 4.4.4. Since we may parametrize ∂D, with counterclockwise orientation, by

$$\varphi(t) = (a\cos(t), b\sin(t)),$$

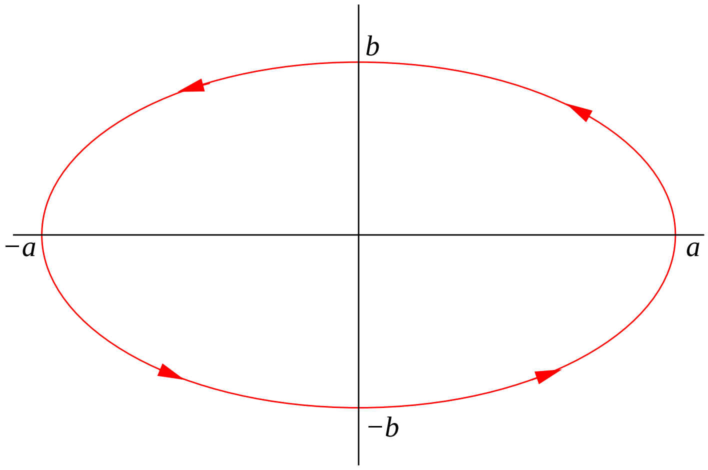

Figure 4.4.4 The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with counterclockwise orientation

$0 \le t \le 2\pi$, we have

$$\begin{aligned} A &= \frac{1}{2}\int_{\partial D} x dy - y dx \\ &= \frac{1}{2}\int_0^{2\pi} (-b\sin(t), a\cos(t)) \cdot (-a\sin(t), b\cos(t) dt \\ &= \frac{1}{2}\int_0^{2\pi} (ab\sin^2(t) + ab\cos^2(t)) dt \\ &= \frac{ab}{2}\int_0^{2\pi} dt \\ &= \left(\frac{ab}{2}\right)(2\pi) \\ &= \pi ab. \end{aligned}$$

Problems

1. Let D be the closed rectangle in $\mathbb{R}^2$ with vertices at $(0,0)$, $(2,0)$, $(2,4)$, and $(0,4)$, with boundary ∂D oriented counterclockwise. Use Green's theorem to evaluate the following line integrals.

   (a) $\displaystyle\int_{\partial D} 2xy dx + 3x^2 dy$ (b) $\displaystyle\int_{\partial D} y dx + x dy$

2. Let D be the triangle in $\mathbb{R}^2$ with vertices at $(0,0)$, $(2,0)$, and $(0,4)$, with boundary ∂D oriented counterclockwise. Use Green's theorem to evaluate the following line integrals.

   (a) $\displaystyle\int_{\partial D} 2xy^2 dx + 4x dy$ (b) $\displaystyle\int_{\partial D} y dx + x dy$

   (c) $\displaystyle\int_{\partial D} y dx - x dy$

3. Use Green's theorem to find the area of a circle of radius r.
4. Use Green's theorem to find the area of the region D enclosed by the hypocycloid
$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$
where $a > 0$. Note that we may parametrize this curve using
$$\varphi(t) = (a\cos^3(t), a\sin^3(t)),$$
$0 \leq t \leq 2\pi$.
5. Use Green's theorem to find the area of the region enclosed by one "petal" of the curve parametrized by
$$\varphi(t) = (\sin(2t)\cos(t), \sin(2t)\sin(t)).$$
6. Find the area of the region enclosed by the cardioid parametrized by
$$\varphi(t) = ((2 + \cos(t))\cos(t), (2 + \cos(t))\sin(t)),$$
$0 \leq t \leq 2\pi$.
7. Verify (4.4.23), thus completing the proof of Green's theorem.
8. Suppose the vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ with coordinate functions $p = F_1(x, y)$ and $q = F_2(x, y)$ is C^1 on an open set containing the Type III region D. Moreover, suppose F is the gradient of a scalar function $f : \mathbb{R}^2 \to \mathbb{R}$.
   (a) Show that
$$\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} = 0$$
   for all points (x, y) in D.
   (b) Use Green's theorem to show that
$$\int_{\partial D} p dx + q dy = 0,$$
   where ∂D is the boundary of D with counterclockwise orientation.
9. How many ways do you know to calculate the area of a circle?
10. Who was George Green?
11. Explain how Green's theorem is a generalization of the Fundamental Theorem of Integral Calculus.
12. Let $b > a$, let C_1 be the circle of radius b centered at the origin, and let C_2 be the circle of radius a centered at the origin. If D is the annular region between C_1 and C_2 and F is a C^1 vector field with coordinate functions $p = F_1(x, y)$ and $q = F_2(x, y)$, show that
$$\int\int_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx dy = \int_{C_1} p dx + q dy + \int_{C_2} p dx + q dy,$$
where C_1 is oriented in the counterclockwise direction and C_2 is oriented in the clockwise direction. (Hint: Decompose D into Type III regions D_1, D_2, D_3, and D_4, each with boundary oriented counterclockwise, as shown in Figure 4.4.5.)

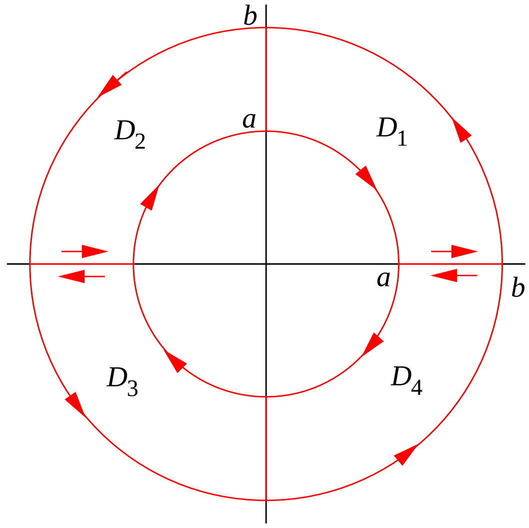

Figure 4.4.5 Decomposition of an annulus into regions of Type III