The Calculus of Functions

of

Several Variables

Section 4.3

Line Integrals

We will motivate the mathematical concept of a line integral through an initial discussion of the physical concept of work.

Work

If a force of constant magnitude F is acting in the direction of motion of an object along a line, and the object moves a distance d along this line, then we call the quantity Fd the *work* done by the force on the object. More generally, if the vector $\mathbf{F}$ represents a constant force acting on an object as it moves along a displacement vector $\mathbf{d}$, then

$$\mathbf{F} \cdot \frac{\mathbf{d}}{\|\mathbf{d}\|} \tag{4.3.1}$$

is the magnitude of $\mathbf{F}$ in the direction of motion (see Figure 4.3.1) and we define

$$\left(\mathbf{F} \cdot \frac{\mathbf{d}}{\|\mathbf{d}\|}\right)\|\mathbf{d}\| = \mathbf{F} \cdot \mathbf{d} \tag{4.3.2}$$

to be the work done by $\mathbf{F}$ on the object when it is displaced by $\mathbf{d}$.

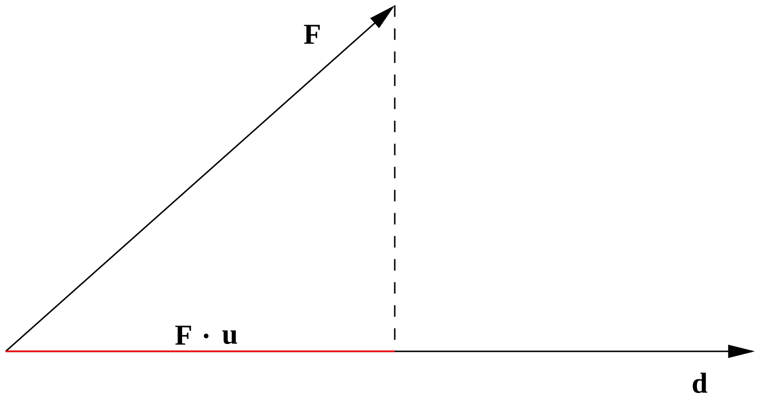

Figure 4.3.1 Magnitude of $\mathbf{F}$ in the direction of $\mathbf{d}$ is $\mathbf{F} \cdot \mathbf{u}$, where $\mathbf{u} = \dfrac{\mathbf{d}}{\|\mathbf{d}\|}$

We now generalize the formulation of work in (4.3.2) to the situation where an object P moves along some curve C subject to a force which depends continuously on position (but does not depend on time). Specifically, we represent the force by a continuous vector field, say, $F : \mathbb{R}^n \to \mathbb{R}^n$, and we suppose P moves along a curve C which has a smooth

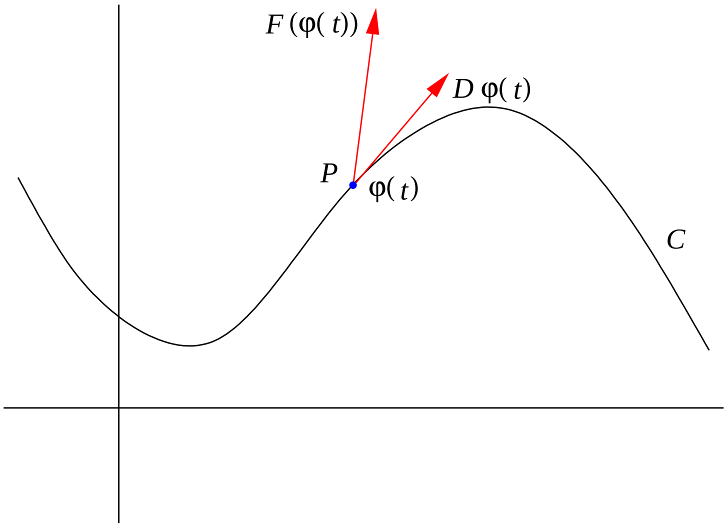

Figure 4.3.2 Object P moving along a curve C subject to a force F

parametrization $\varphi : I \to \mathbb{R}^n$, where $I = [a, b]$. See Figure 4.3.2. To approximate the work done by F as P moves from $\varphi(a)$ to $\varphi(b)$ along C, we first divide I into m equal subintervals of length

$$\Delta t = \frac{b-a}{m}$$

with endpoints $t_0 = a < t_1 < t_2 < \cdots < t_m = b$. Now at time t_k, $k = 0, 1, \ldots, m-1$, P is moving in the direction of $D\varphi(t_k)$ at a speed of $\|D\varphi(t_k)\|$, and so will move a distance of approximately $\|D\varphi(t_k)\|\Delta t$ over the time interval $[t_k, t_{k+1}]$. Thus we may approximate the work done by F as P moves from $\varphi(t_k)$ to $\varphi(t_{k+1})$ by the work done by the force $F(\varphi(t_k))$ in moving P along the displacement vector $D\varphi(t_k)\Delta t$, which is a vector of length $\|D\varphi(t_k)\|\Delta t$ in the direction of $D\varphi(t_k)$. That is, if we let W_k denote the work done by F as P moves from $\varphi(t_k)$ to $\varphi(t_{k-1})$, then

$$W_k \approx F(\varphi(t_k)) \cdot D\varphi(t_k)\Delta t. \tag{4.3.3}$$

If we let W denote the total work done by F as P moves along C, then we have

$$W = \sum_{k=0}^{m-1} W_k = \sum_{k=0}^{m-1} F(\varphi(t_k)) \cdot D\varphi(t_k)\Delta t. \tag{4.3.4}$$

As m increases, we should expect the approximation in (4.3.4) to approach W. Moreover, since $F(\varphi(t)) \cdot D\varphi(t)$ is a continuous function of t and the sum in (4.3.4) is a left-hand rule approximation for the definite integral of $F(\varphi(t)) \cdot D\varphi(t)$ over the interval $[a, b]$, we should have

$$W = \lim_{m\to\infty} \sum_{k=0}^{m-1} F(\varphi(t_k)) \cdot D\varphi(t_k)\Delta t = \int_a^b F(\varphi(t)) \cdot D\varphi(t)dt. \tag{4.3.5}$$

Example Suppose an object moves along the curve C parametrized by $\varphi(t) = (t, t^2)$, $-1 \leq t \leq 1$, subject to the force $F(x, y) = (y, x)$. Then the work done by F as the object moves from $\varphi(-1) = (-1, 1)$ to $\varphi(1) = (1, 1)$ is

$$\begin{aligned} W &= \int_{-1}^{1} F(\varphi(t)) \cdot D\varphi(t) dt \\ &= \int_{-1}^{1} F(t, t^2) \cdot (1, 2t) dt \\ &= \int_{-1}^{1} (t^2, t) \cdot (1, 2t) dt \\ &= \int_{-1}^{1} 3t^2 dt \\ &= t^3 \Big|_{-1}^{1} \\ &= 2. \end{aligned}$$

Example The function $\psi(t) = \left(\frac{t}{2}, \frac{t^2}{4}\right)$, $-2 \leq t \leq 2$, is also a smooth parametrization of the curve C in the previous example. Using the same force function F, we have

$$\begin{aligned} \int_{-2}^{2} F(\psi(t)) \cdot D\psi(t) dt &= \int_{-2}^{2} \left(\frac{t^2}{4}, \frac{t}{2}\right) \cdot \left(\frac{1}{2}, \frac{t}{2}\right) dt \\ &= \int_{-2}^{2} \frac{3t^2}{8} \, dt \\ &= \frac{t^3}{8}\Big|_{-2}^{2} \\ &= 2. \end{aligned}$$

This is the result we should expect: as long as the curve is traversed only once, the work done by a force when an object moves along the curve should depend only on the curve and not on any particular parametrization of the curve.

We need to verify the previous statement in general before we can state our definition of the line integral. Note that in these two examples, $\psi(t) = \varphi\left(\frac{t}{2}\right)$. In other words, $\psi(t) = \varphi(g(t))$, where $g(t) = \frac{t}{2}$ for $-2 \leq t \leq 2$. In general, if $\varphi(t)$, for t in an interval $[a, b]$, and $\psi(t)$, for t in an interval $[c, d]$, are both smooth parametrizations of a curve C such that every point on C corresponds to exactly one point in I and exactly one point in J, then there exists a differentiable function g which maps J onto I such that $\psi(t) = \varphi(g(t))$. Defining such a g is straightforward: given any t in $[c, d]$, find the unique value s in $[a, b]$ such that $\varphi(s) = \psi(t)$ (such a value s has to exist since C is the image of both ψ and φ). Then $g(t) = s$. Proving that g is differentiable is not as easy, and we will not provide a proof here. However, assuming that g is differentiable, it follows that for any continuous vector field F,

$$\begin{aligned}\int_c^d F(\psi(t))\cdot D\psi(t)dt &= \int_c^d F(\varphi(g(t))\cdot D(\varphi\circ g)(t))dt \\ &= \int_c^d F(\varphi(g(t))\cdot D\varphi(g(t))g'(t)dt. \qquad (4.3.6)\end{aligned}$$

Now if we let

$$\begin{aligned}u &= g(t),\\ du &= g'(t)dt,\end{aligned}$$

in (4.3.6), then

$$\int_c^d F(\psi(t))\cdot D\psi(t)dt = \int_a^b F(\varphi(u))\cdot D\varphi(u)dt \qquad (4.3.7)$$

if $g(c) = a$ and $g(d) = b$ (that is, $\varphi(a) = \psi(c)$ and $\varphi(b) = \psi(d)$, and

$$\int_c^d F(\psi(t))\cdot D\psi(t)dt = \int_b^a F(\varphi(u))\cdot D\varphi(u)dt = -\int_b^a F(\varphi(u))\cdot D\varphi(u)dt \qquad (4.3.8)$$

if $g(c) = b$ and $g(d) = a$ (that is, $\varphi(a) = \psi(d)$ and $\varphi(b) = \psi(c)$. Note that the second case occurs only if ψ parametrizes C in the reverse direction of φ, in which case we say ψ is an *orientation reversing reparametrization* of φ. In the first case, that is, when $\varphi(a) = \psi(c)$ and $\varphi(b) = \psi(d)$, we say ψ is an *orientation preserving reparametrization* of φ. Our results in (4.3.7) and (4.3.8) then correspond to the physical notion that the work done by a force in moving an object along a curve is the negative of the work done by the force in moving the object along the curve in the opposite direction. From now on, when referring to a curve C, we will assume some orientation, or direction, has been specified. We will then use $-C$ to refer to the curve consisting of the same set of points as C, but with the reverse orientation.

Line integrals

Now that we know that, except for direction, the value of the integral involved in computing work does not depend on the particular parametrization of the curve, we may state a formal mathematical definition.

Definition Suppose C is a curve in $\mathbb{R}^n$ with smooth parametrization $\varphi : I \to \mathbb{R}^n$, where $I = [a, b]$ is an interval in $\mathbb{R}$. If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector field, then we define the *line integral* of F along C, denoted

$$\int_C F\cdot ds,$$

by

$$\int_C F\cdot ds = \int_a^b F(\varphi(t))\cdot D\varphi(t)dt. \qquad (4.3.9)$$

As a consequence of our previous remarks, we have the following result.

Proposition Using the notation of the definition,

$$\int_C F \cdot ds$$

depends only on the curve C and its orientation, not on the parametrization φ. Moreover,

$$\int_{-C} F \cdot ds = -\int_C F \cdot ds. \tag{4.3.10}$$

Example Let C be the unit circle centered at the origin in $\mathbb{R}^2$, oriented in the counter-clockwise direction, and let

$$F(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) = \frac{1}{x^2+y^2}(-y,x).$$

To compute the line integral of F along C, we first need to find a smooth parametrization of C. One such parametrization is

$$\varphi(t) = (\cos(t), \sin(t))$$

for $0 \leq t \leq 2\pi$. Then

$$\begin{aligned}
\int_C F \cdot ds &= \int_0^{2\pi} F(\cos(t), \sin(t)) \cdot (-\sin(t), \cos(t))dt \\
&= \int_0^{2\pi} \frac{1}{\cos^2(t)+\sin^2(t)}(-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t))dt \\
&= \int_0^{2\pi} (\sin^2(t) + \cos^2(t))dt \\
&= \int_0^{2\pi} dt \\
&= 2\pi.
\end{aligned}$$

Note that $\psi(t) = (\sin(t), \cos(t))$, $0 \leq t \leq 2\pi$, parametrizes $-C$, from which we can calculate

$$\begin{aligned}
\int_{-C} F \cdot ds &= \int_0^{2\pi} F(\sin(t), \cos(t)) \cdot (\cos(t), -\sin(t))dt \\
&= \int_0^{2\pi} \frac{1}{\sin^2(t)+\cos^2(t)}(-\cos(t), \sin(t)) \cdot (\cos(t), -\sin(t))dt \\
&= \int_0^{2\pi} (-\cos^2(t) - \sin^2(t))dt \\
&= -\int_0^{2\pi} dt \\
&= -2\pi,
\end{aligned}$$

in agreement with the previous proposition.

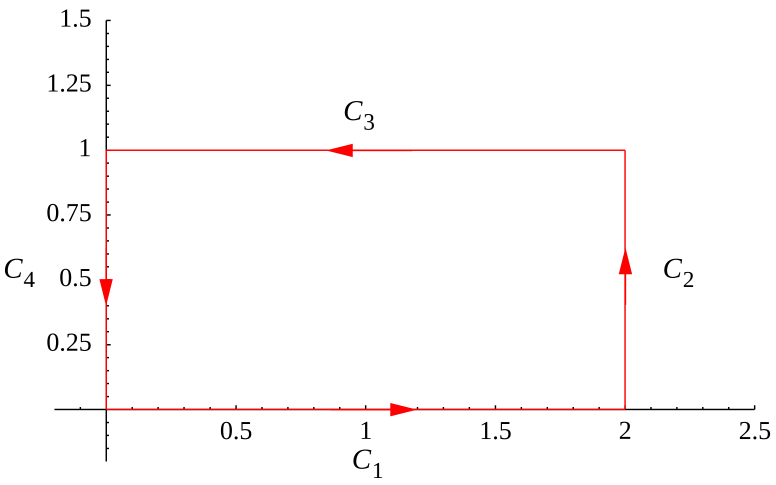

Figure 4.3.3 Rectangle with counterclockwise orientation

A *piecewise smooth* curve is one which may be decomposed into a finite number of curves, each of which has a smooth parametrization. If C is a piecewise smooth curve composed of the union of the curves C_1, C_2, $\ldots$, C_m, then we may extend the definition of the line integral to C by defining

$$\int_C F \cdot ds = \int_{C_1} F \cdot ds + \int_{C_2} F \cdot ds + \cdots + \int_{C_m} F \cdot ds. \tag{4.3.11}$$

The next example illustrates this procedure.

Example Let C be the rectangle in $\mathbb{R}^2$ with vertices at $(0,0)$, $(2,0)$, $(2,1)$, and $(0,1)$, oriented in the counterclockwise direction, and let $F(x,y) = (y^2, 2xy)$. If we let C_1, C_2, C_3, and C_4 be the four sides of C, as labeled in Figure 4.3.3, then we may parametrize C_1 by

$$\alpha(t) = (t, 0),$$

$0 \leq t \leq 2$, C_2 by

$$\beta(t) = (2, t),$$

$0 \leq t \leq 1$, C_3 by

$$\gamma(t) = (2 - t, 1),$$

$0 \leq t \leq 2$, and C_4 by

$$\delta(t) = (0, 1 - t),$$

$0 \leq t \leq 1$. Then

$$\begin{aligned}\int_C F \cdot ds &= \int_{C_1} F \cdot ds + \int_{C_2} F \cdot ds + \int_{C_3} F \cdot ds + \int_{C_4} F \cdot ds \\ &= \int_0^2 F(t,0) \cdot (1,0)dt + \int_0^1 F(2,t) \cdot (0,1)dt + \int_0^2 F(2-t,1) \cdot (-1,0)dt\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_0^1 F(0,1-t)\cdot(0,-1)dt \\
&= \int_0^2 (0,0)\cdot(1,0)dt + \int_0^1 (t^2,4t)\cdot(0,1)dt + \int_0^2 (1,4-2t)\cdot(-1,0)dt \\
&\quad + \int_0^1 ((1-t)^2,0)\cdot(0,-1)dt \\
&= \int_0^2 0dt + \int_0^1 4tdt + \int_0^2 (-1)dt + \int_0^1 0dt \\
&= 2t^2\Big|_0^1 - 2 \\
&= 2-2 \\
&= 0.
\end{aligned}
$$

Note that it would be slightly simpler to parametrize $-C_3$ and $-C_4$, using

$$\varphi(t) = (1,t),$$

$0 \leq t \leq 2$, and

$$\psi(t) = (t,0),$$

$0 \leq t \leq 1$, respectively, than to parametrize C_3 and C_4 directly. We would then evaluate

$$\int_C F\cdot ds = \int_{C_1} F\cdot ds + \int_{C_2} F\cdot ds - \int_{-C_3} F\cdot ds - \int_{-C_4} F\cdot ds.$$

A note on notation

Suppose C is a smooth curve in $\mathbb{R}^n$, parametrized by $\varphi : I \to \mathbb{R}^n$, where $I = [a,b]$, and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field. Our notation for the line integral of F along C comes from letting $s = \varphi(t)$, from which we have

$$\frac{ds}{dt} = D\varphi(t),$$

which we may write, symbolically, as

$$ds = D\varphi(t)dt.$$

Now suppose $\varphi_1, \varphi_2, \ldots, \varphi_n$ and $F_1, F_2, \ldots, F_n$ are the component functions of φ and F, respectively. If we let

$$
\begin{aligned}
x_1 &= \varphi_1(t), \\
x_2 &= \varphi_2(t), \\
&\ \vdots \\
x_n &= \varphi_n(t),
\end{aligned}
$$

then we may write

$$\begin{aligned}\int_C F\cdot ds &= \int_a^b F(\varphi(t))\cdot D\varphi(t)dt\\ &= \int_a^b F(x_1(t),x_2(t),\ldots,x_n(t))\cdot(\varphi_1'(t),\varphi_2'(t),\ldots,\varphi_n'(t))dt\\ &= \int_a^b (F_1(x_1(t),x_2(t),\ldots,x_n(t))\varphi_1'(t)+F_2(x_1(t),x_2(t),\ldots,x_n(t))\varphi_2'(t))+\cdots\\ &\qquad + F_n(x_1(t),x_2(t),\ldots,x_n(t))\varphi_n'(t))dt\\ &= \int_a^b F_1(x_1(t),x_2(t),\ldots,x_n(t))\varphi_1'(t)dt+\int_a^b F_2(x_1(t),x_2(t),\ldots,x_n(t))\varphi_2'(t)dt\\ &\qquad +\cdots+\int_a^b F_n(x_1(t),x_2(t),\ldots,x_n(t))\varphi_n'(t)dt. \end{aligned}\tag{4.3.12}$$

Suppressing the dependence on t, writing dx_k for $\varphi_k'(t)dt$, $k=1,2,\ldots,n$, and using only a single integral sign, we may rewrite (4.3.12) as

$$\int_C F_1(x_1,x_2,\ldots,x_n)dx_1+F_2(x_1,x_2,\ldots,x_n)dx_2+\cdots+F_n(x_1,x_2,\ldots,x_n)dx_n. \tag{4.3.13}$$

This is a common, and useful, notation for a line integral.

Example We will evaluate

$$\int_C ydx+xdy+z^2dz,$$

where C is the part of a helix in $\mathbb{R}^3$ with parametric equations

$$\begin{aligned}x&=\cos(t),\\ y&=\sin(t),\\ z&=t,\end{aligned}$$

$0\le t\le 2\pi$. Note that this is equivalent to evaluating

$$\int_C F\cdot ds,$$

where $F:\mathbb{R}^3\to\mathbb{R}^3$ is the vector field $F(x,y,z)=(y,x,z^2)$. We have

$$\begin{aligned}\int_C ydx+xdy+z^2dz &= \int_0^{2\pi}(\sin(t)(-\sin(t))+\cos(t)\cos(t)+t^2)dt\\ &= \int_0^{2\pi}(\cos^2(t)-\sin^2(t)+t^2)dt\\ &= \int_0^{2\pi}(\cos(2t)+t^2)dt\\ &= \frac{1}{2}\sin(2t)\Big|_0^{2\pi}+\frac{1}{3}t^3\Big|_0^{2\pi}\\ &= \frac{8\pi^3}{3}.\end{aligned}$$

Gradient fields

Recall that if $f : \mathbb{R}^n \to \mathbb{R}$ is C^1, then ∇f is a continuous vector field on $\mathbb{R}^n$. Suppose $\varphi : I \to \mathbb{R}^n$, $I = [a, b]$, is a smooth parametrization of a curve C. Then, using the chain rule and the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_C \nabla f \cdot ds &= \int_a^b \nabla f(\varphi(t)) \cdot D\varphi(t)dt \\ &= \int_a^b \frac{d}{dt} f(\varphi(t))dt \\ &= f(\varphi(t))\Big|_a^b \\ &= f(\varphi(b)) - f(\varphi(a)).\end{aligned}$$

Theorem If $f : \mathbb{R}^n \to \mathbb{R}$ is C^1 and $\varphi : I \to \mathbb{R}^n$, $I = [a, b]$, is a smooth parametrization of a curve C, then

$$\int_C \nabla f \cdot ds = f(\varphi(b)) - f(\varphi(a)). \tag{4.3.14}$$

Note that (4.3.14) shows that the value of a line integral of a gradient vector field depends only on the starting and ending points of the curve, not on which particular path is taken between these two points. Moreover, (4.3.14) provides a simple means for evaluating a line integral if the given vector field can be identified as the gradient of a scalar valued function. Another interesting consequence is that if the beginning and ending points of C are the same, that is, if $\mathbf{v} = \varphi(a) = \varphi(b)$, then

$$\int_C \nabla f \cdot ds = f(\varphi(b)) - f(\varphi(b)) = f(\mathbf{v}) - f(\mathbf{v}) = 0. \tag{4.3.15}$$

We call such curves *closed curves*. In words, the line integral of a gradient vector field is 0 along any closed curve.

Example If $F(x, y) = (y, x)$, then

$$F(x, y) = \nabla f(x, y),$$

where $f(x, y) = xy$. Hence, for example, for any smooth curve C starting at $(-1, 1)$ and ending at $(1, 1)$ we have

$$\int_C F \cdot ds = f(1, 1) - f(-1, 1) = 1 + 1 = 2.$$

Note that this agrees with the result in our first example above, where C was the part of the parabola $y = x^2$ extending from $(-1, 1)$ to $(1, 1)$.

Example If $f(x, y) = xy^2$, then

$$\nabla f(x, y) = (y^2, 2xy).$$

If C is the rectangle in $\mathbb{R}^2$ with vertices at $(0,0)$, $(2,0)$, $(2,1)$, and $(0,1)$, then, since C is a closed curve,

$$\int_C y^2dx + 2xydy = 0,$$

in agreement with an earlier example. Similarly, if E is the unit circle in $\mathbb{R}^2$ centered at the origin, then we know that

$$\int_E y^2dx + 2xydy = 0,$$

with no need for further computations.

In physics, a force field F is said to be *conservative* if the work done by F in moving an object between any two points depends only on the points, and not on the path used between the two points. In particular, we have shown that if F is the gradient of some scalar function f, then F is a conservative force field. Under certain conditions on the domain of F, the converse is true as well. That is, under certain conditions, if F is a conservative force field, then there exists a scalar function f such that $F = \nabla f$. Problem 9 explores one such situation in which this is true. The function f is then known as a *potential function*.

Problems

1. For each of the following, compute the line integral $\displaystyle\int_C F \cdot ds$ for the given vector field F and curve C parametrized by φ.
   (a) $F(x,y) = (xy, 3x)$, $\varphi(t) = (t^2, t)$, $0 \le t \le 2$
   (b) $F(x,y) = \left(\dfrac{y}{x^2+y^2}, \dfrac{x}{x^2+y^2}\right)$, $\varphi(t) = (\cos(t), \sin(t))$, $0 \le t \le 2\pi$
   (c) $F(x,y) = (3x - 2y, 4x^2y)$, $\varphi(t) = (t^3, t^2)$, $-2 \le t \le 2$
   (d) $F(x,y,z) = (xyz, 3xy^2, 4z)$, $\varphi(t) = (3t, t^2, 4t^3)$, $0 \le t \le 4$
2. Let C be the circle of radius 2 centered at the origin in $\mathbb{R}^2$, with counterclockwise orientation. Evaluate the following line integrals.
   (a) $\displaystyle\int_C 3xdx + 4ydy$ (b) $\displaystyle\int_C 8xydx + 4x^2dy$
3. Let C be the part of a helix in $\mathbb{R}^3$ parametrized by $\varphi(t) = (\cos(2t), \sin(2t), t)$, $0 \le t \le 2\pi$. Evaluate the following line integrals.
   (a) $\displaystyle\int_C 3xdx + 4ydy + zdz$ (b) $\displaystyle\int_C yzdx + xzdy + xydz$
4. Let C be the rectangle in $\mathbb{R}^2$ with vertices at $(-1,1)$, $(2,1)$, $(2,3)$, and $(-1,3)$, with counterclockwise orientation. Evaluate the following line integrals.
   (a) $\displaystyle\int_C x^2ydx + (3y+x)dy$ (b) $\displaystyle\int_C 2xydx + x^2dy$

5. Let C be the ellipse in $\mathbb{R}^2$ with equation
$$\frac{x^2}{4} + \frac{y^2}{9} = 1,$$
with counterclockwise orientation. Evaluate $\int_C F \cdot ds$ for $F(x, y) = (4y, 3x)$.

6. Let C be the upper half of the circle of radius 3 centered at the origin in $\mathbb{R}^2$, with counterclockwise orientation. Evaluate the following line integrals.
   (a) $\int_C 3y\,dx$ (b) $\int_C 4x\,dy$

7. Evaluate
$$\int_C \frac{x}{x^2 + y^2}\,dx + \frac{y}{x^2 + y^2}\,dy,$$
where C is any curve which starts at $(1, 0)$ and ends at $(2, 3)$.

8. (a) Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 vector field which is the gradient of a scalar function $f : \mathbb{R}^n \to \mathbb{R}$. If F_k is the kth coordinate function of F, $k = 1, 2, \ldots, n$, show that
$$\frac{\partial}{\partial x_j} F_i(x_1, x_2, \ldots, x_n) = \frac{\partial}{\partial x_i} F_j(x_1, x_2, \ldots, x_n)$$
for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

   (b) Show that although
$$\int_C x\,dx + xy\,dy = 0$$
for every circle C in $\mathbb{R}^2$ with center at the origin, nevertheless $F(x, y) = (x, xy)$ is not the gradient of any scalar function $f : \mathbb{R}^n \to \mathbb{R}$.

   (c) Let
$$F(x, y) = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$
for all (x, y) in the set $S = \{(x, y) : (x, y) \neq (0, 0)\}$. Let F_1 and F_2 be the coordinate functions of F. Show that
$$\frac{\partial}{\partial y} F_1(x, y) = \frac{\partial}{\partial x} F_2(x, y)$$
for all (x, y) in S, even though F is not the gradient of any scalar function . (Hint: For the last part, show that
$$\int_C F \cdot ds = 2\pi,$$
where C is the unit circle centered at the origin.)

9. Suppose $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous vector field with the property that for any curve C,

$$\int_C F \cdot ds$$

depends only on the endpoints of C. That is, if C_1 and C_2 are any two curves with the same endpoints P and Q, then

$$\int_{C_1} F \cdot ds = \int_{C_2} F \cdot ds.$$

(a) Show that

$$\int_C F \cdot ds = 0$$

for any closed curve C.

(b) Let F_1 and F_2 be the coordinate functions of F. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \int_C F \cdot ds,$$

where C is any curve which starts at $(0, 0)$ and ends at (x, y). Show that

$$\frac{\partial}{\partial y} f(x, y) = F_2(x, y).$$

(Hint: In evaluating $f(x, y)$, consider the curve C from $(0, 0)$ to (x, y) which consists of the horizontal line from $(0, 0)$ to $(x, 0)$ followed by the vertical line from $(x, 0)$ to (x, y).)

(c) Show that $\nabla f = F$.