*The Calculus of Functions*

*of*

*Several Variables*

## Section 4.1

## Geometry, Limits, and Continuity

In this chapter we will treat the general case of a function mapping $\mathbb{R}^m$ to $\mathbb{R}^n$. Since the cases $m = 1$ and $n = 1$ have been handled in previous chapters, our emphasis will be on the higher dimensional cases, most importantly when $m$ and $n$ are 2 or 3. We will begin in this section with some basic terminology and definitions.

### Parametrized surfaces

If $f : \mathbb{R}^m \to \mathbb{R}^n$ has domain $D$, we call the set $S$ of all points $\mathbf{y}$ in $\mathbb{R}^n$ for which $\mathbf{y} = f(\mathbf{x})$ for some $\mathbf{x}$ in $D$ the *image* of $f$. That is,

$$S = \{f(\mathbf{x}) : \mathbf{x} \in D\}, \tag{4.1.1}$$

which is the same as what we have previously called the range of $f$. If $m = 1$, $S$ is a curve as defined in Section 2.1. If $m > 1$ and $n > m$, then we call $S$ an *m-dimensional* surface in $\mathbb{R}^n$. If we let $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ and $(y_1, y_2, \ldots, y_n) = f(x_1, x_2, \ldots, x_m)$, then, for $k = 1, 2, \ldots, n$, we call the function $f_k : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f_k(x_1, x_2, \ldots, x_m) = y_k$$

the *k-th coordinate function* of $f$. We call the system of equations

$$\begin{aligned} y_1 &= f_1(x_1, x_2, \ldots, x_m), \\ y_2 &= f_2(x_1, x_2, \ldots, x_m), \\ \vdots &= \qquad \vdots \\ y_n &= f_n(x_1, x_2, \ldots, x_m), \end{aligned} \tag{4.1.2}$$

a *parametrization* of the surface $S$. Note that $f_k$ is the type of function we studied in Chapter 3. On the other hand, if we fix values of $x_i$ for all $i \neq k$, then the function $\varphi_k : \mathbb{R} \to \mathbb{R}^n$ defined by

$$\varphi_k(t) = f(x_1, x_2, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_m) \tag{4.1.3}$$

is of the type we studied we Chapter 2. In particular, for each $k = 1, 2, \ldots, n$, $\varphi_k$ parametrizes a curve which lies on the surface $S$. The following examples illustrate how the latter remark is useful when trying to picture a parametrized surface $S$.

**Example** Consider $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$f(s, t) = (t\cos(s), t\sin(s), t)$$

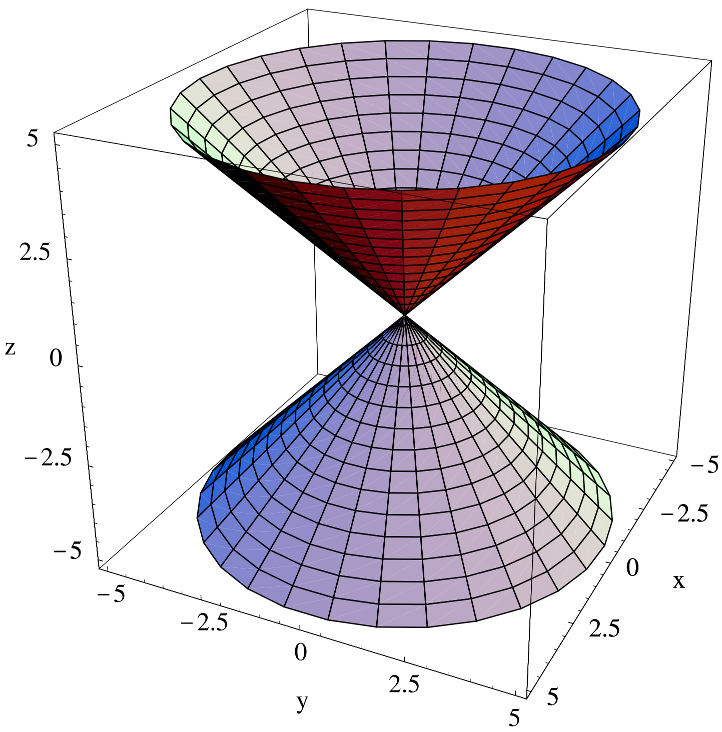


Figure 4.1.1 Cone parametrized by $f(s,t) = (t\cos(s), t\sin(s), t)$

for $0 \le s \le 2\pi$ and $-\infty < t < \infty$. The image of $f$ is the surface $S$ in $\mathbb{R}^3$ parametrized by the equations

$$\begin{aligned} x &= t\cos(s), \\ y &= t\sin(s), \\ z &= t. \end{aligned}$$

Note that for a fixed value of $t$, the function

$$\varphi_1(s) = (t\cos(s), t\sin(s), t)$$

parametrizes a circle of radius $t$ on the plane $z = t$ with center at $(0,0,t)$. On the other hand, for a fixed value of $s$, the function

$$\varphi_2(t) = (t\cos(s), t\sin(s), t) = t(\cos(s), \sin(s), 1)$$

parametrizes a line through the origin in the direction of the vector $(\cos(s), \sin(s), 1$. Hence the surface $S$ is a cone in $\mathbb{R}^3$, part of which is shown in Figure 4.1.1. Notice how the surface was drawn by plotting the curves corresponding to fixed values of $s$ and $t$ (that is, the curves parametrized by $\varphi_1$ and $\varphi_2$), and then filling in the resulting curvilinear "rectangles."

**Example** For a fixed $a > 0$, consider the function $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$f(s,t) = (a\cos(s)\sin(t), a\sin(s)\sin(t), a\cos(t))$$

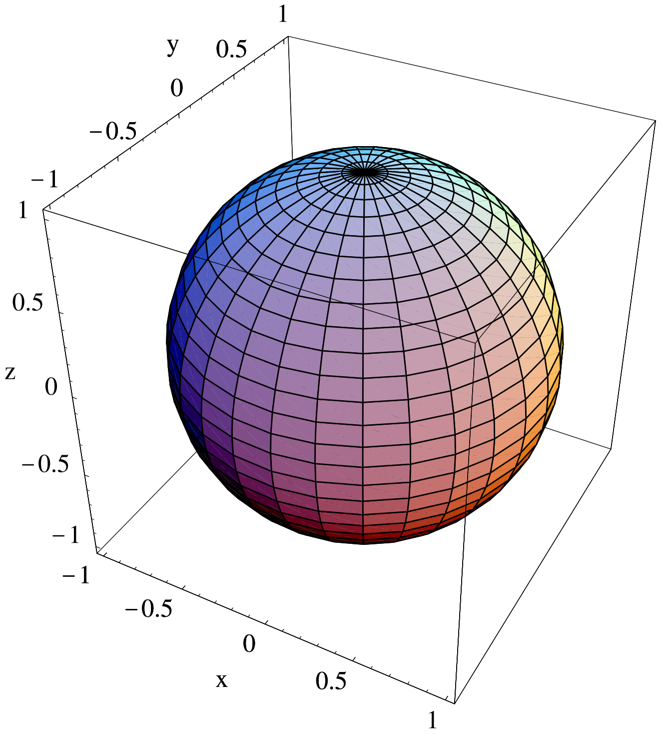


Figure 4.1.2 Unit sphere parametrized by $f(s,t) = (\cos(s)\sin(t), \sin(s)\sin(t), \cos(t))$

for $0 \le s \le 2\pi$ and $0 \le t \le \pi$. The image of $f$ is the surface $S$ in $\mathbb{R}^3$ parametrized by the equations

$$\begin{aligned} x &= a\cos(s)\sin(t), \\ y &= a\sin(s)\sin(t), \\ z &= a\cos(t). \end{aligned} \tag{4.1.4}$$

Note that these are the equations for the spherical coordinate change of variables discussed in Section 3.7, with $\rho = a$, $\theta = s$, and $\varphi = t$. Since $a$ is fixed while $s$ varies from 0 to $2\pi$ and $t$ varies from 0 to $\pi$, it follows that $S$ is a sphere of radius $a$ with center $(0, 0, 0)$. Figure 4.1.2 displays $S$ when $a = 1$. If we had not previously studied spherical coordinates, we could reach this conclusion about $S$ as follows. First note that

$$\begin{aligned} x^2 + y^2 + z^2 &= a^2\cos^2(s)\sin^2(t) + a^2\sin^2(s)\sin^2(t) + a^2\cos^2(t) \\ &= a^2\sin^2(t)(\cos^2(s) + \sin^2(s)) + a^2\cos^2(t) \\ &= a^2(\sin^2(t) + \cos^2(t)) \\ &= a^2, \end{aligned}$$

from which it follows that every point of $S$ lies on the sphere of radius $a$ centered at the origin. Now for a fixed value of $t$,

$$\varphi_1(s) = (a\cos(s)\sin(t), a\sin(s)\sin(t), a\cos(t))$$

parametrizes a circle in the plane $z = a\cos(t)$ with center $(0, 0, a\cos(t))$ and radius $a\sin(t)$. As $t$ varies from 0 to $\pi$, these circles vary from a circle in the $z = a$ plane with center

$(0,0,a)$ and radius 0 (when $t=0$) to a circle in the $xy$-plane with center $(0,0,0)$ and radius $a$ (when $t=\frac{\pi}{2}$) to a circle in the $z=-a$ plane with center $(0,0,-a)$ and radius 0 (when $t=\pi$). In other words, the circles fill in all the "lines of latitude" of the sphere from the "North Pole" to the "South Pole," and hence $S$ is all of the sphere. One may also show that the functions

$$\varphi_2(t)=(a\cos(s)\sin(t),a\sin(s)\sin(t),a\cos(t))$$

parametrize the "lines of longitude" of $S$ as $s$ varies from 0 to $2\pi$. Both the lines of "latitude" and "longitude" are visible in Figure 4.2.2.

**Example** Suppose $0<b<a$ and define $f:\mathbb{R}^2\to\mathbb{R}^3$ by

$$f(s,t)=((a+b\cos(t))\cos(s),(a+b\cos(t))\sin(s),b\sin(t))$$

for $0\le s\le 2\pi$ and $0\le t\le 2\pi$. The image of $f$ is the surface $T$ parametrized by the equations

$$\begin{aligned} x&=(a+b\cos(t))\cos(s),\\ y&=(a+b\cos(t))\sin(s),\\ z&=b\sin(t). \end{aligned}$$

Note that for a fixed value of $t$,

$$\varphi_1(s)=((a+b\cos(t))\cos(s),(a+b\cos(t))\sin(s),b\sin(t))$$

parametrizes a circle in the plane $z=b\sin(t)$ with center $(0,0,b\sin(t)$ and radius $a+b\cos(t)$. In particular, when $t=0$, we have a circle in the $xy$-plane with center $(0,0,0)$ and radius $a+b$; when $t=\frac{\pi}{2}$, we have a circle on the plane $z=b$ with center $(0,0,b)$ and radius $a$; when $t=\pi$, we have a circle on the $xy$-plane with center $(0,0,0)$ and radius $a-b$; when $t=\frac{3\pi}{2}$, we have a circle on the $z=-b$ plane with center $(0,0,-b)$ and radius $a$; and when $t=2\pi$, we are back to a circle in the $xy$-plane with center $(0,0,0)$ and radius $a+b$. For fixed values of $s$, the curves parametrized by

$$\varphi_2(t)=((a+b\cos(t))\cos(s),(a+b\cos(t))\sin(s),b\sin(t))$$

are not identified as easily. However, some particular cases are illuminating. When $s=0$, we have a circle in the $xz$-plane with center $(a,0,0)$ and radius $b$; when $s=\frac{\pi}{2}$, we have a circle in the $yz$-plane with center $(0,a,0)$ and radius $b$; when $s=\pi$, we have a circle in the $xz$-plane with center $(-a,0,0)$ and radius $b$; when $s=\frac{3\pi}{2}$, we have a circle in the $yz$-plane with center $(0,-a,0)$ and radius $b$; and when $t=2\pi$, we are back to a circle in the $xz$-plane with center $(a,0,0)$ and radius $b$. Putting all this together, we see that $T$ is a *torus*, the surface of a doughnut shaped object. Figure 4.1.3 shows one such torus, the case $a=3$ and $b=1$.

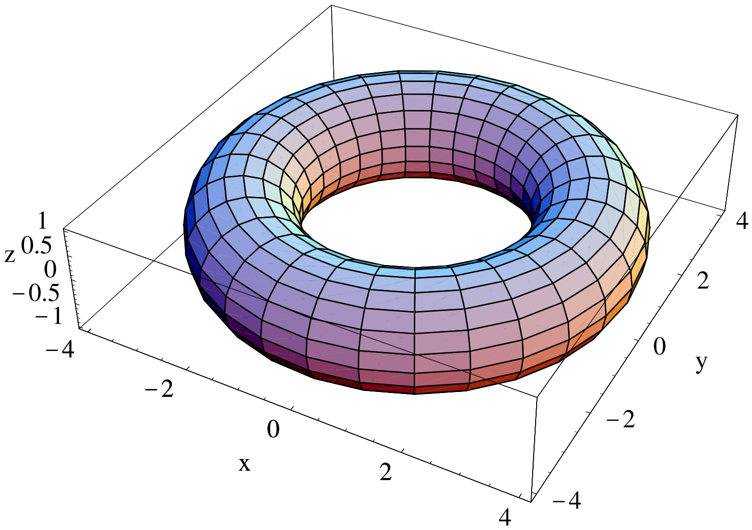


Figure 4.1.3 A torus: $f(s,t) = ((3+\cos(t))\cos(s), (3+\cos(t))\sin(s), \sin(t))$

**Vector fields**

We call a function $f : \mathbb{R}^n \to \mathbb{R}^n$, that is, a function for which the domain and range space have the same dimension, a *vector field*. We have seen a few examples of such functions already. For example, the change of variable functions in Section 3.7 were of this type. Also, given a function $g : \mathbb{R}^n \to \mathbb{R}$, the gradient of $g$,

$$\nabla g(\mathbf{x}) = \left( \frac{\partial}{\partial x_1} g(\mathbf{x}), \frac{\partial}{\partial x_2} g(\mathbf{x}), \ldots, \frac{\partial}{\partial x_n} g(\mathbf{x}) \right),$$

is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. As we saw in our discussion of gradient vector fields in Section 3.2, a plot showing the vectors $f(\mathbf{x})$ at each point in a rectangular grid provides a useful geometric view of a vector field $f$.

**Example** Consider the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$f(\mathbf{x}) = -\frac{\mathbf{x}}{\|\mathbf{x}\|^2}$$

for all $\mathbf{x} \neq \mathbf{0}$. Note that $f(\mathbf{x})$ is a vector of length

$$\left\| \frac{\mathbf{x}}{\|\mathbf{x}\|^2} \right\| = \frac{\|\mathbf{x}\|}{\|\mathbf{x}\|^2} = \frac{1}{\|\mathbf{x}\|}$$

pointing in the direction opposite that of $\mathbf{x}$. If $n = 2$, the coordinate functions of $f$ are

$$f_1(x_1, x_2) = -\frac{x_1}{x_1^2 + x_2^2}$$

and

$$f_2(x_1, x_2) = -\frac{x_2}{x_1^2 + x_2^2}.$$

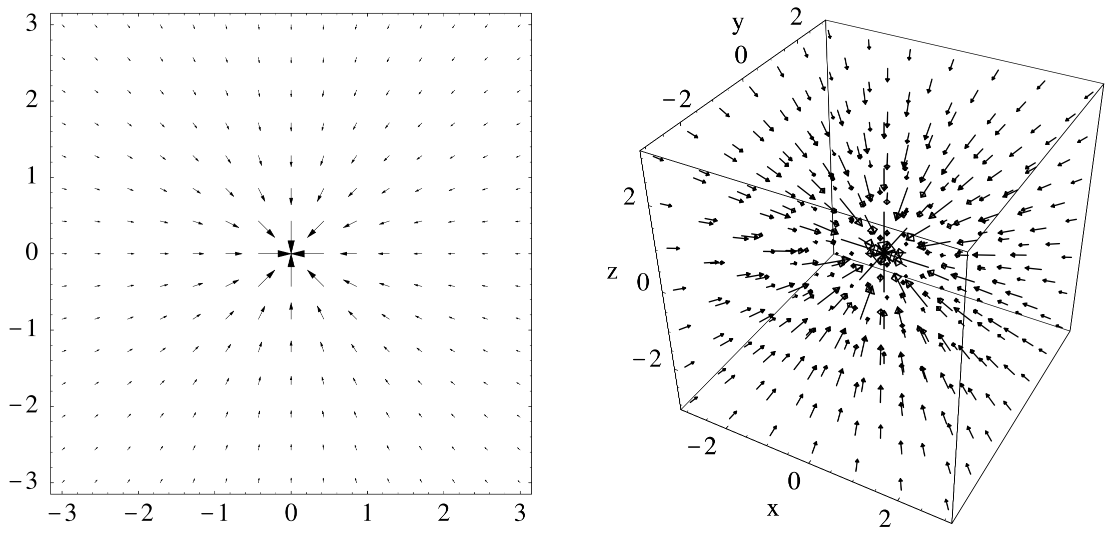


Figure 4.1.4 Vector field $f(\mathbf{x}) = \dfrac{\mathbf{x}}{\|\mathbf{x}\|^2}$ for $n = 2$ and $n = 3$

Figure 4.1.4 shows a plot of the vectors $f(\mathbf{x})$ for this case, drawn on a grid over the rectangle $[-3, 3] \times [-3, 3]$, and for the case $n = 3$, using the cube $[-3, 3] \times [-3, 3] \times [-3, 3]$. Note that these plots do not show the vectors $f(\mathbf{x})$ themselves, but vectors which have been scaled proportionately so they do not overlap one another.

### Limits and continuity

The definitions of limits and continuity for functions $f : \mathbb{R}^m \to \mathbb{R}^n$ follow the familiar pattern.

**Definition** Let $\mathbf{a}$ be a point in $\mathbb{R}^m$ and let $O$ be the set of all points in the open ball of radius $r > 0$ centered at $\mathbf{a}$ except $\mathbf{a}$. That is,

$$O = \{\mathbf{x} : \mathbf{x} \in B^m(\mathbf{a}, r), \mathbf{x} \neq \mathbf{a}\}.$$

Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is defined for all $\mathbf{x}$ in $O$. We say the *limit* of $f(\mathbf{x})$ as $\mathbf{x}$ approaches $\mathbf{a}$ is $\mathbf{L}$, written $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{L}$, if for every sequence of points $\{\mathbf{x}_k\}$ in $O$,

$$\lim_{k\to\infty} f(\mathbf{x}_k) = \mathbf{L} \tag{4.1.5}$$

whenever $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{a}$.

In Section 2.1 we saw that a sequence of points in $\mathbb{R}^n$ has a limit if and only if the individual coordinates of the points in the sequence each have a limit. The following proposition is an immediate consequence.

**Proposition** If $f_k : \mathbb{R}^m \to \mathbb{R}$, $k = 1, 2, \ldots, n$, is the $k$th coordinate function of $f : \mathbb{R}^m \to \mathbb{R}^n$, then

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = (L_1, L_2, \ldots, L_n)$$

if and only if

$$\lim_{\mathbf{x} \to \mathbf{a}} f_k(\mathbf{x}) = L_k$$

for $k = 1, 2, \ldots, n$.

In other words, the computation of limits for functions $f : \mathbb{R}^m \to \mathbb{R}^n$ reduces to the familiar problem of computing limits of real-valued functions, as we discussed in Section 3.1.

**Example** If

$$f(x, y, z) = (x^2 - 3yz, 4xz),$$

a function from $\mathbb{R}^3$ to $\mathbb{R}^2$, then, for example,

$$\lim_{(x,y,z) \to (1,-2,3)} f(x, y, z) = \left( \lim_{(x,y,z) \to (1,-2,3)} (x^2 - 3yz), \lim_{(x,y,z) \to (1,-2,3)} 4xz \right) = (19, 12).$$

**Definition** Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is defined for all $\mathbf{x}$ in some open ball $B^n(\mathbf{a}, r)$, $r > 0$. We say $f$ is *continuous* at $\mathbf{a}$ if $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$.

The following result is an immediate consequence of the previous proposition.

**Proposition** If $f_k : \mathbb{R}^m \to \mathbb{R}$, $k = 1, 2, \ldots, n$, is the $k$th coordinate function of $f : \mathbb{R}^m \to \mathbb{R}^n$, then $f$ is continuous at a point $\mathbf{a}$ if and only if $f_k$ is continuous at $\mathbf{a}$ for $k = 1, 2, \ldots, n$.

In other words, checking for continuity for a function $f : \mathbb{R}^m \to \mathbb{R}^n$ reduces to checking the continuity of real-valued functions, a familiar problem from Section 3.1.

**Example** The function

$$f(x, y) = (3\sin(x + y), 4x^2 y)$$

has coordinate functions

$$f_1(x, y) = 3\sin(x + y)$$

and

$$f_2(x, y) = 4x^2 y.$$

Since, from our results in Section 3.1, both $f_1$ and $f_2$ are continuous at every point in $\mathbb{R}^2$, it follows that $f$ is continuous at every point in $\mathbb{R}^2$.

**Definition** We say a function $f : \mathbb{R}^m \to \mathbb{R}^n$ is *continuous* on an open set $U$ if $f$ is continuous at every point $\mathbf{u}$ in $U$.

**Example** We may restate the conclusion of the previous example by saying that

$$f(x, y) = (3\sin(x + y), 4x^2 y)$$

is continuous on $\mathbb{R}^2$.

**Problems**

1. For each of the following, plot the surface parametrized by the given function.
   (a) $f(s,t) = (t^2\cos(s), t^2\sin(s), t^2)$, $0 \le s \le 2\pi$, $0 \le t \le 3$
   (b) $f(u,v) = (3\cos(u)\sin(v), \sin(u)\sin(v), 2\cos(v))$, $0 \le u \le 2\pi$, $0 \le v \le \pi$
   (c) $g(s,t) = ((4+2\cos(t))\cos(s), (4+2\cos(t))\sin(s), 2\sin(t))$, $0 \le s \le 2\pi$, $0 \le t \le 2\pi$
   (d) $f(s,t) = ((5+2\cos(t))\cos(s), 2(5+2\cos(t))\sin(s), \sin(t))$, $0 \le s \le 2\pi$, $0 \le t \le 2\pi$
   (e) $h(u,v) = (\sin(v), (3+\cos(v))\cos(u), (3+\cos(v))\sin(u))$, $0 \le u \le 2\pi$, $0 \le v \le 2\pi$
   (f) $g(s,t) = (s, s^2+t^2, t)$, $-2 \le s \le 2$, $-2 \le t \le 2$
   (g) $f(x,y) = (y\cos(x), y, y\sin(x))$, $0 \le x \le 2\pi$, $-5 \le y \le 5$
2. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ and we define $F : \mathbb{R}^2 \to \mathbb{R}^3$ by $F(s,t) = (s,t,f(s,t))$. Describe the surface parametrized by $F$.
3. Find a parametrization for the surface that is the graph of the function $f(x,y) = x^2 + y^2$.
4. Make plots like those in Figure 4.1.4 for each of the following vector fields. Experiment with the rectangle used for the grid, as well as with the number of vectors drawn.
   (a) $f(x,y) = (y, -x)$
   (b) $g(x,y) = (y, -\sin(x))$
   (c) $f(u,v) = (v, u - u^3 - v)$
   (d) $f(x,y) = (x(1-y^2) - y, x)$
   (e) $f(x,y,z) = \left(10(y-x), 28x - y - xz, -\dfrac{8}{3}z + xy\right)$
   (f) $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}(x,y,z)$
   (g) $g(u,v,w) = -\dfrac{1}{(u-1)^2+(v-2)^2+(w-1)^2}(u-1, v-2, w-1)$
5. Find the set of points in $\mathbb{R}^2$ for which the vector field
$$f(x,y) = \left(4x\sin(x-y), \frac{4x+3y}{2x-y}\right)$$
   is continuous.
6. For which points in $\mathbb{R}^n$ is the vector field
$$f(\mathbf{x}) = \frac{\mathbf{x}}{\log(\|\mathbf{x}\|)}$$
   a continuous function?