The Calculus of Functions

of

Several Variables

Section 3.7

Change of Variables in Integrals

One of the basic techniques for evaluating an integral in one-variable calculus is substitution, replacing one variable with another in such a way that the resulting integral is of a simpler form. Although slightly more subtle in the case of two or more variables, a similar idea provides a powerful technique for evaluating definite integrals.

Linear change of variables

We will present the main idea through an example. Let

$$D = \{(x,y) : 9x^2 + 4y^2 \leq 36\},$$

the region inside the ellipse which intersects the x-axis at $(-2,0)$ and $(2,0)$ and the y-axis at $(0,-3)$ and $(0,3)$. To find the area of D, we evaluate

$$\int\int_D dxdy = \int_{-2}^{2}\int_{-\frac{3}{2}\sqrt{4-x^2}}^{\frac{3}{2}\sqrt{4-x^2}} dydx = \int_{-2}^{2} 3\sqrt{4-x^2}\, dx = 6\pi,$$

where the final integral may be evaluated using the substitution $x = 2\sin(\theta)$ or by noting that

$$\int_{-2}^{2} \sqrt{4-x^2}\, dx$$

is one-half of the area of a circle of radius 2. Alternatively, suppose we write the equation of the ellipse as

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

and make the substitution $x = 2u$ and $y = 3v$. Then $u = \frac{x}{2}$ and $v = \frac{y}{3}$, so if (x,y) is a point in D, then

$$u^2 + v^2 = \frac{x^2}{4} + \frac{y^2}{9} \leq 1.$$

That is, if (x,y) is a point in D, then (u,v) is a point in the unit disk

$$E = \{(u,v) : u^2 + v^2 \leq 1\}.$$

Conversely, if (u,v) is a point in E, then

$$\frac{x^2}{4} + \frac{y^2}{9} = \frac{4u^2}{4} + \frac{9v^2}{9} = u^2 + v^2 \leq 1,$$

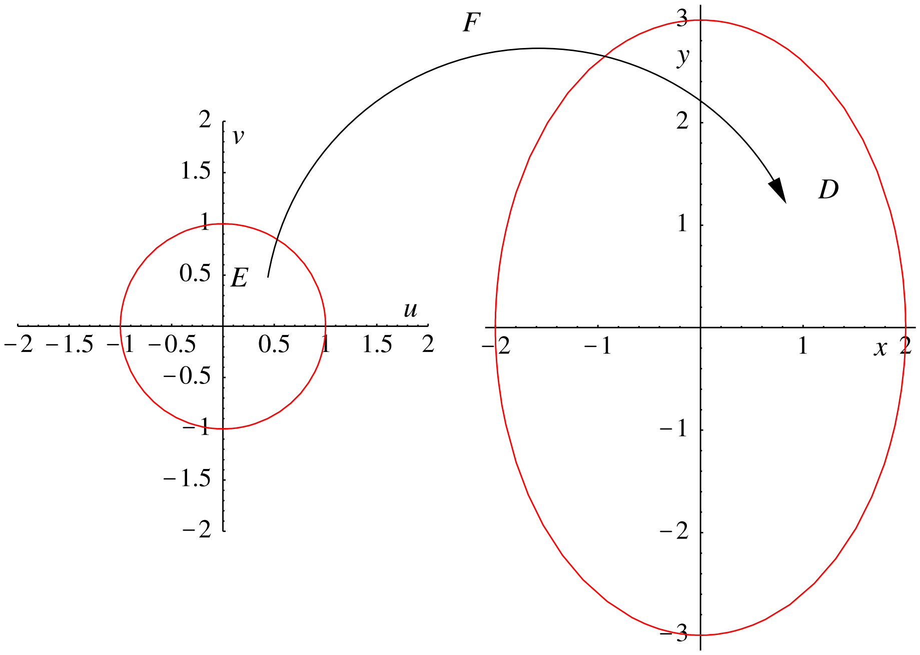

Figure 3.7.1 F maps E onto D

so (x, y) is a point in D. Thus the function $F(u, v) = (2u, 3v)$ takes the region E, a closed disk of radius 1, and stretches it onto the region D (as shown in Figure 3.7.1). However, note that even though every point in E corresponds to exactly one point in D, and, conversely, every point in D corresponds to exactly on point in E, nevertheless E and D do not have the same area. To see how F changes area, consider what it does to the unit square S with sides $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. The area of S is 1, but F maps S onto a rectangle R with sides

$$F(1, 0) = (2, 0)$$

and

$$F(0, 1) = (0, 3)$$

and area 6. This a special case of a general fact we saw in Section 1.6: the linear function F, with associated matrix

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

maps the unit square S onto a parallelogram R with area

$$|\det(M)| = 6.$$

The important fact for us here is that 1 unit of area in the uv-plane corresponds to 6 units of area in the xy-plane. Hence the area of D will be 6 times the area of E. That is,

$$\int\int_D dxdy = \int\int_E |\det(M)| dudv = \int\int_E 6dudv = 6\int\int_E dudv = 6\pi,$$

where the final integral is simply the area inside a circle of radius 1.

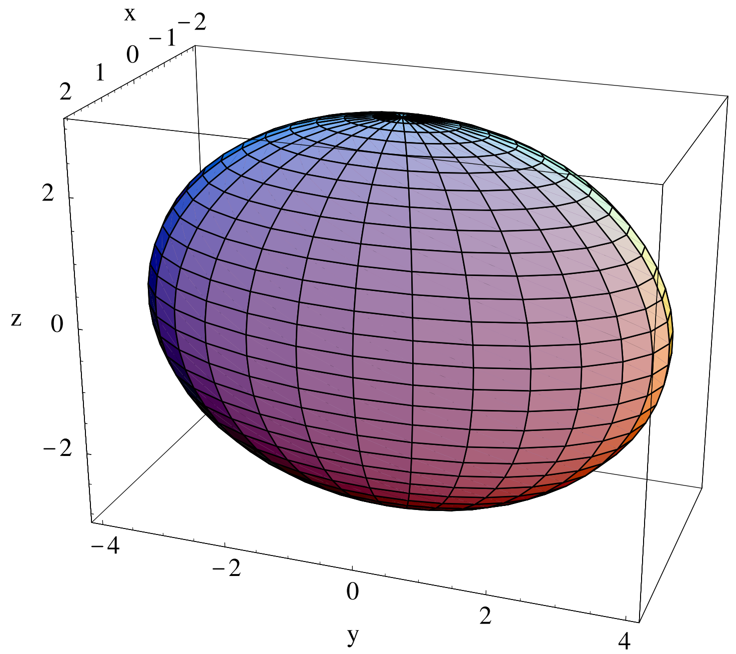

Figure 3.7.2 The ellipsoid $\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = 1$

These ideas provide the background for a proof of the following theorem.

Theorem Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous on a an open set U containing the closed bounded set D. Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a linear function, M is an $n \times n$ matrix such that $F(\mathbf{u}) = M\mathbf{u}$, and $\det(M) \neq 0$. If F maps the region E onto the region D and we define the change of variables

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = M \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix},$$

then

$$\begin{aligned} &\int\int\cdots\int_D f(x_1,x_2,\ldots,x_n)dx_1dx_2\cdots dx_n \\ &\qquad = \int\int\cdots\int_E f(F(u_1,u_2,\ldots,u_n))|\det(M)|du_1du_2\cdots du_n. \end{aligned} \tag{3.7.1}$$

Example Let D be the region in $\mathbb{R}^3$ bounded by the ellipsoid with equation

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = 1.$$

See Figure 3.7.2. If we make the change of variables $x = 2u$, $y = 4v$, and $z = 3w$, that is,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix},$$

then, for any (x, y, z) in D, we have

$$u^2 + v^2 + w^2 = \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} \leq 1.$$

That is, if (x, y, z) lies in D, then the corresponding (u, v, w) lies in the closed unit ball $E = \bar{B}^3((0,0,0), 1)$. Conversely, if (u, v, w) lies in E, then

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = \frac{4u^2}{4} + \frac{16v^2}{16} + \frac{9w^2}{9} = u^2 + v^2 + w^2 \leq 1,$$

so (x, y, z) lies in D. Hence, the change of variables $F(u, v, w) = (2u, 4v, 3w)$ maps E onto D. Now

$$\det \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 24,$$

so if V is the volume of D, then

$$V = \int\int\int_D dxdydz = \int\int\int_E 24dudvdw = 24\int\int\int_E dudvdw = 24\left(\frac{4\pi}{3}\right) = 32\pi,$$

where we have used the fact that the volume of a sphere of radius 1 is $\frac{4\pi}{3}$ to evaluate the final integral.

Nonlinear change of variables

Without going into the technical details, we will indicate how to proceed when the change of variables is not linear. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous on a an open set U containing the closed bounded set D and $F : \mathbb{R}^n \to \mathbb{R}^n$ maps a closed bounded region E of $\mathbb{R}^n$ onto D so that every point of D corresponds to exactly one point of E. Writing $F(\mathbf{u}) = (F_1(\mathbf{u}), F_2(\mathbf{u}), \ldots, F_n(\mathbf{u}))$, we will assume that F_1, F_2, $\ldots$, and F_n are all differentiable on an open set W containing E. Although we will not study this type of function until Chapter 4, the natural candidate for the derivative of F is the matrix whose ith row is $\nabla F_i(\mathbf{u})$. Letting $x_i = F_i(u_1, u_2, \ldots, u_n)$, $i = 1, 2, \ldots, n$, we denote this matrix, called the *Jacobian matrix* of F,

$$\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(u_1, u_2, \ldots, u_n)}. \tag{3.7.2}$$

Explicitly,

$$\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(u_1, u_2, \ldots, u_n)} = \begin{bmatrix} \frac{\partial}{\partial u_1}F_1(\mathbf{u}) & \frac{\partial}{\partial u_2}F_1(\mathbf{u}) & \cdots & \frac{\partial}{\partial u_n}F_1(\mathbf{u}) \\ \frac{\partial}{\partial u_1}F_2(\mathbf{u}) & \frac{\partial}{\partial u_2}F_2(\mathbf{u}) & \cdots & \frac{\partial}{\partial u_n}F_2(\mathbf{u}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial u_1}F_n(\mathbf{u}) & \frac{\partial}{\partial u_2}F_n(\mathbf{u}) & \cdots & \frac{\partial}{\partial u_n}F_n(\mathbf{u}) \end{bmatrix}. \tag{3.7.3}$$

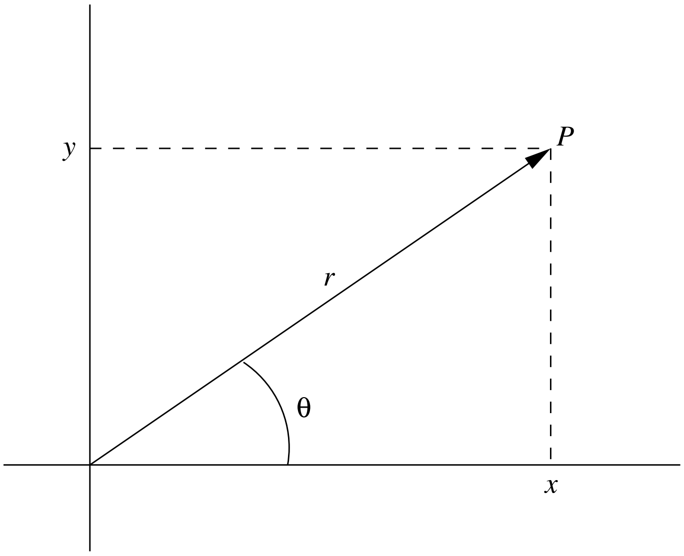

Figure 3.7.3 Polar and Cartesian coordinates for a point P

We shall see in Chapter 4 that

$$\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(u_1, u_2, \ldots, u_n)}$$

is the matrix for the linear part of the best affine approximation to F at $(u_1, u_2, \ldots, u_n)$. Hence, for sufficiently small rectangles, the factor by which F changes the area of a rectangle when it maps it to a region will be approximately

$$\left|\det \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(u_1, u_2, \ldots, u_n)}\right|. \tag{3.7.4}$$

One may then show that, analogous to (3.7.1), we have

$$\begin{aligned} &\int \cdots \int \int_D f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \cdots dx_n \\ &\quad = \int \cdots \int \int_E f(F(u_1, u_2, \ldots, u_n)) \left|\det \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(u_1, u_2, \ldots, u_n)}\right| du_1 du_2 \cdots du_n. \end{aligned} \tag{3.7.5}$$

Note that (3.7.5) is just (3.7.1) with the matrix M replaced by the Jacobian of F.

We will now look at two very useful special cases of the preceding result. See Problems 22 and 23 for a third special case.

Polar coordinates

As an alternative to describing the location of a point P in the plane using its Cartesian coordinates (x, y), we may locate the point using r, the distance from P to the origin, and θ, the angle between the vector from $(0, 0)$ to P and the positive x-axis, measured in the counterclockwise direction from 0 to 2π (see Figure 3.7.3). That is, if P has Cartesian coordinates (x, y), with $x \neq 0$, we may define its *polar coordinates* (r, θ) by specifying that

$$r = \sqrt{x^2 + y^2} \tag{3.7.6}$$

and

$$\tan(\theta) = \frac{y}{x}, \tag{3.7.7}$$

where we take $0 \le \theta \le \pi$ if $y \ge 0$ and $\pi < \theta < 2\pi$ if $y < 0$. If $x = 0$, we let $\theta = \frac{\pi}{2}$ if $y > 0$ and $\theta = \frac{3\pi}{2}$ if $y < 0$. For $(x, y) = (0, 0)$, $r = 0$ and θ could have any value, and so is undefined. Conversely, if a point P has polar coordinates (r, θ), then

$$x = r\cos(\theta) \tag{3.7.8}$$

and

$$y = r\sin(\theta). \tag{3.7.9}$$

Note that the choice of the interval $[0, 2\pi)$ for the values of θ is not unique, with any interval of length 2π working as well. Although $[0, 2\pi)$ is the most common choice for values of θ, it is sometimes useful to use $(-\pi, \pi)$ instead.

Example If a point P has Cartesian coordinates $(-1, 1)$, then its polar coordinates are $\left(\sqrt{2}, \frac{3\pi}{4}\right)$.

Example A point with polar coordinates $\left(3, \frac{\pi}{6}\right)$ has Cartesian coordinates $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$.

In our current context, we want to think of the polar coordinate mapping

$$(x, y) = F(r, \theta) = (r\cos(\theta), r\sin(\theta)) \tag{3.7.10}$$

as a change of variables between the $r\theta$-plane and the xy-plane. This mapping is particularly useful for us because it maps rectangular regions in the $r\theta$-plane onto circular regions in the xy-plane. For example, for any $a > 0$, F maps the rectangular region

$$E = \{(r, \theta) : 0 \le r \le a, 0 \le \theta < 2\pi\}$$

in the $r\theta$-plane onto the closed disk

$$D = \bar{B}^2((0, 0), a) = \{(x, y) : x^2 + y^2 \le a\}$$

in the xy-plane (see Figure 3.7.5 below for an example). More generally, for any $0 \le \alpha < \beta < 2\pi$, F maps the rectangular region

$$E = \{(r, \theta) : 0 \le r \le a, \alpha \le \theta < \beta\}$$

in the $r\theta$-plane onto a region D in the xy-plane which is the sector of the closed disk $\bar{B}^2((0, 0), a)$ which lies between radii of angles α and β (see Figure 3.7.4). Another basic example is an annulus: for any $0 < a < b$, F maps the rectangular region

$$E = \{(r, \theta) : a \le r \le b, 0 \le \theta < 2\pi\}$$

in the $r\theta$-plane onto the annulus

$$D = \{(x, y) : a \le x^2 + y^2 \le b\}$$

in the xy-plane. Figure 3.7.6 illustrates this mapping for the upper half of an annulus.

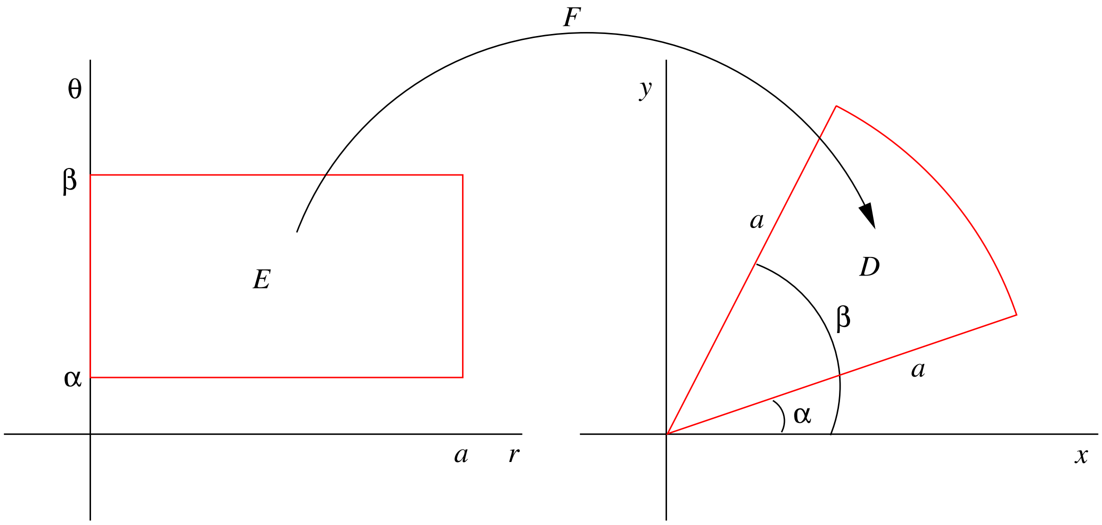

Figure 3.7.4 Polar coordinate change of variables

Example Let V be the volume of the region which lies beneath the paraboloid with equation $z = 4 - x^2 - y^2$ and above the xy-plane. In Section 3.6, we saw that

$$V = \int\int_D (4 - x^2 - y^2)dxdy = 8\pi,$$

where

$$D = \{(x, y) : x^2 + y^2 \leq 4\}.$$

The use of polar coordinates greatly simplifies the evaluation of this integral. With the polar coordinate change of variables

$$x = r\cos(\theta)$$

and

$$y = r\sin(\theta),$$

the closed disk D in the xy-plane corresponds to the closed rectangle

$$E = \{(r, \theta) : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$$

in the $r\theta$-plane (see Figure 3.7.5). Note that in describing E we have allowed $\theta = 2\pi$, but this has no affect on our outcome since a line has no area in $\mathbb{R}^2$. Moreover, if we let $f(x, y) = 4 - x^2 - y^2$, then

$$\begin{aligned} f(F(r,\theta)) &= f(r\cos(\theta), r\sin(\theta)) \\ &= 4 - r^2\cos^2(\theta) - r^2\sin(\theta) \\ &= 4 - r^2(\cos^2(\theta) + \sin^2(\theta)) \\ &= 4 - r^2, \end{aligned}$$

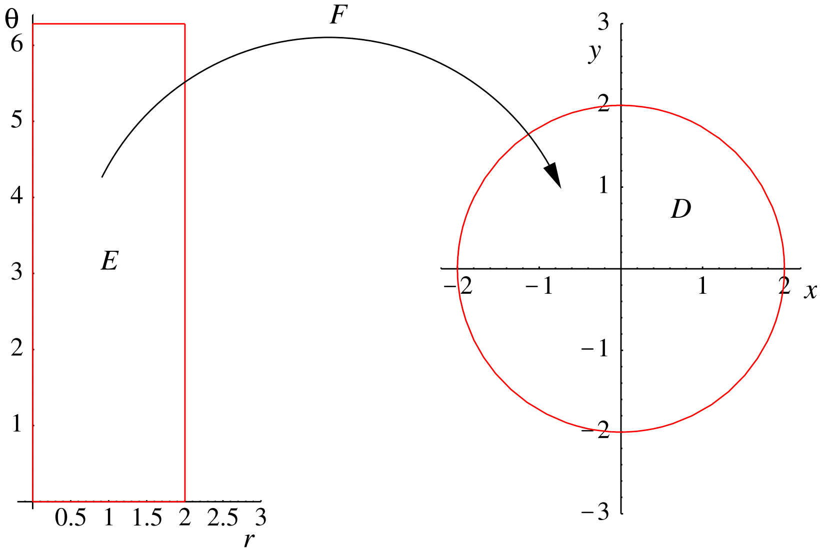

Figure 3.7.5 Polar coordinate change of variables maps $[0,2]\times[0,2\pi]$ to $\bar{B}^2((0,0),2)$

which also follows from the fact that $r^2 = x^2 + y^2$. Now

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{bmatrix} \dfrac{\partial}{\partial r} r\cos(\theta) & \dfrac{\partial}{\partial \theta} r\cos(\theta) \\ \dfrac{\partial}{\partial r} r\sin(\theta) & \dfrac{\partial}{\partial \theta} r\sin(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}, \tag{3.7.11}$$

so

$$\det \frac{\partial(x,y)}{\partial(r,\theta)} = r\cos^2(\theta) + r\sin^2(\theta) = r(\cos^2(\theta)+\sin^2(\theta)) = r. \tag{3.7.12}$$

Hence, using (3.7.5), we have

$$\begin{aligned} \int\!\!\int_D (4-x^2-y^2)dxdy &= \int\!\!\int_E (4-r^2)\left|\det\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta \\ &= \int_0^2\int_0^{2\pi}(4-r^2)rd\theta dr \\ &= \int_0^2 2\pi(4r-r^3)dr \\ &= 2\pi\left(2r^2-\frac{r^4}{4}\right)\Bigg|_0^2 \\ &= 2\pi(8-4) \\ &= 8\pi. \end{aligned}$$

Example Suppose D is the part of the region between the circles with equations $x^2+y^2 = 1$ and $x^2+y^2=9$ which lies above the x-axis. That is,

$$D = \{(x,y) : 1 \le x^2+y^2 \le 9, x \ge 0\}.$$

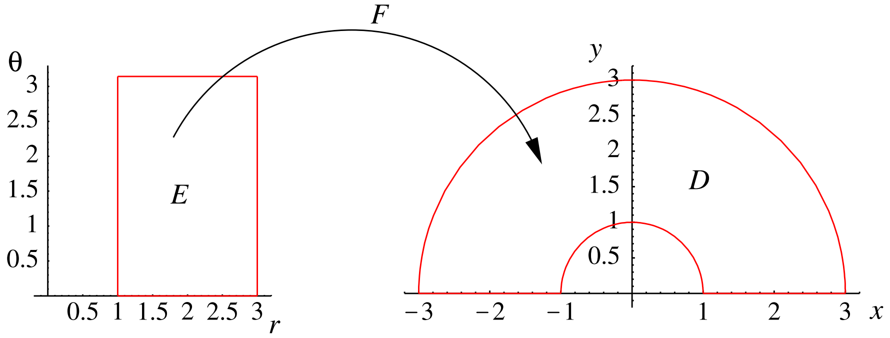

Figure 3.7.6 Polar coordinates map $[1,3] \times [0,\pi]$ to top half of an annulus

We wish to evaluate

$$\int\int_D e^{-(x^2+y^2)} dxdy.$$

Under the polar coordinate change of variables

$$x = r\cos(\theta)$$

and

$$y = r\sin(\theta),$$

the annular region D corresponds to the closed rectangle

$$E = \{(r,\theta) : 1 \le r \le 3, 0 \le \theta \le \pi\},$$

as illustrated in Figure 3.7.6. Moreover, $x^2 + y^2 = r^2$ and, as we saw in the previous example,

$$\left|\det \frac{\partial(x,y)}{\partial(r,\theta)}\right| = r.$$

Hence

$$\begin{aligned}
\int\int_D e^{-(x^2+y^2)} dxdy &= \int\int_E re^{-r^2} drd\theta \\
&= \int_1^3 \int_0^\pi re^{-r^2} d\theta dr \\
&= \int_1^3 \pi re^{-r^2} dr \\
&= -\frac{\pi}{2} e^{-r^2}\Big|_1^3 \\
&= \frac{\pi}{2}(e^{-1} - e^{-9}).
\end{aligned}$$

Note that in this case the change of variables not only simplified the region of integration, but also put the function being integrated into a form to which we could apply the Fundamental Theorem of Calculus.

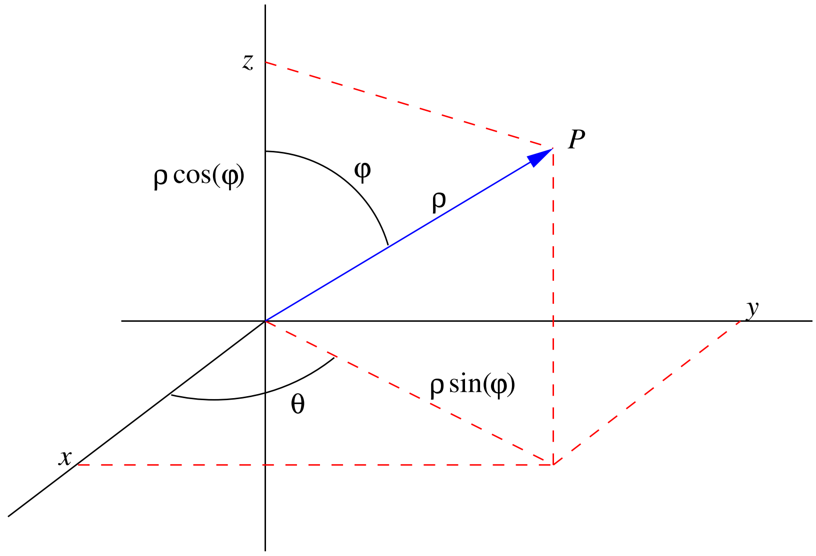

Figure 3.7.7 Spherical and Cartesian coordinates for a point P

Spherical coordinates

Next consider the following extension of polar coordinates to three space: given a point P with Cartesian coordinates (x, y, z), let ρ be the distance from P to the origin, θ be the angle coordinate for the polar coordinates of $(x, y, 0)$ (the projection of P onto the xy-plane), and let φ be the angle between the vector from the origin to P and the positive z-axis, measured from 0 to π. If $x \neq 0$, we have

$$\rho = \sqrt{x^2 + y^2 + z^2}, \tag{3.7.13}$$

$$\tan(\theta) = \frac{y}{x}, \tag{3.7.14}$$

and

$$\cos(\varphi) = \frac{z}{\sqrt{x^2 + y^2 + z^2}}, \tag{3.7.15}$$

where $0 \leq \theta < 2\pi$ and $0 \leq \varphi \leq \pi$. As with polar coordinates, if $x = 0$ we let $\theta = \frac{\pi}{2}$ if $y > 0$, $\theta = \frac{3\pi}{2}$ if $y < 0$, and θ is undefined if $y = 0$. See Figure 3.7.7. Conversely, given a point P with spherical coordinates (ρ, θ, φ), the projection of P onto the xy-plane will have polar coordinate $r = \rho\sin(\varphi)$. Hence the Cartesian coordinates of P are

$$x = \rho\cos(\theta)\sin(\varphi), \tag{3.7.16}$$

$$y = \rho\sin(\theta)\sin(\varphi), \tag{3.7.17}$$

and

$$z = \rho\cos(\varphi). \tag{3.7.18}$$

Example If a point P has Cartesian coordinates $(2, -2, 1)$, then its spherical coordinates satisfy

$$\rho = \sqrt{4 + 4 + 1} = 3,$$

$$\tan(\theta) = \frac{-2}{2} = -1,$$

and

$$\cos(\varphi) = \frac{1}{\sqrt{4+4+1}} = \frac{1}{3}.$$

Hence we have

$$\theta = \frac{7\pi}{4}$$

and

$$\varphi = \cos^{-1}\left(\frac{1}{3}\right) = 1.2310,$$

where we have rounded the value of φ to four decimal places. Hence P has spherical coordinates $\left(3, \frac{7\pi}{4}, 1.2310\right)$.

Example If a point P has spherical coordinates $\left(4, \frac{\pi}{3}, \frac{3\pi}{4}\right)$, then its Cartesian coordinates are

$$x = 4\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{3\pi}{4}\right) = 4\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2},$$

$$y = 4\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{3\pi}{4}\right) = 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{6},$$

and

$$z = 4\cos\left(\frac{3\pi}{4}\right) = 4\left(-\frac{1}{\sqrt{2}}\right) = -2\sqrt{2}.$$

Analogous to our work with polar coordinates, we think of the spherical coordinate mapping

$$(x, y, z) = F(\rho, \theta, \varphi) = (\rho\cos(\theta)\sin(\varphi), \rho\sin(\theta)\sin(\varphi), \rho\cos(\varphi)) \tag{3.7.19}$$

as a change of variables between $\rho\theta\varphi$-space and xyz-space. This mapping is particularly useful for evaluating triple integrals because it maps rectangular regions in $\rho\theta\varphi$-space onto spherical regions in xyz-space. For the most basic example, for any $a > 0$, F maps the rectangular region

$$E = \{(\rho, \theta, \varphi) : 0 \le \rho \le a, 0 \le \theta < 2\pi, 0 \le \varphi \le \pi\}$$

in $\rho\theta\varphi$-space onto the closed ball

$$D = \bar{B}^3((0,0,0), a) = \{(x, y, z) : x^2 + y^2 + z^2 \le a\}$$

in xyz-space. More generally, for any $0 < a < b$, $0 \le \alpha < \beta < 2\pi$, and $0 \le \gamma < \delta \le \pi$, F maps the rectangular region

$$E = \{(\rho, \theta, \varphi) : a \le \rho \le b, \alpha \le \theta < \beta, \gamma \le \varphi \le \delta\}$$

onto a region D in xyz-space which lies between the concentric spheres $S^2((0,0,0),a)$ and $S^2((0,0,0),b)$, and for which the angle θ lies between α and β and the angle φ between γ and δ. For example, if $\alpha = 0$, $\beta = \pi$, $\gamma = 0$, and $\delta = \frac{\pi}{2}$, then D is one-half of the region lying between two concentric hemispheres with radii a and b.

Before using the spherical coordinate change of variable in (3.7.19) to evaluate an integral using (3.7.5), we need to compute the determinate of the Jacobian of F. Now

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \begin{bmatrix} \frac{\partial}{\partial\rho}\rho\cos(\theta)\sin(\varphi) & \frac{\partial}{\partial\theta}\rho\cos(\theta)\sin(\varphi) & \frac{\partial}{\partial\varphi}\rho\cos(\theta)\sin(\varphi) \\ \frac{\partial}{\partial\rho}\rho\sin(\theta)\sin(\varphi) & \frac{\partial}{\partial\theta}\rho\sin(\theta)\sin(\varphi) & \frac{\partial}{\partial\varphi}\rho\sin(\theta)\sin(\varphi) \\ \frac{\partial}{\partial\rho}\rho\cos(\varphi) & \frac{\partial}{\partial\theta}\rho\cos(\varphi) & \frac{\partial}{\partial\varphi}\rho\cos(\varphi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta)\sin(\varphi) & -\rho\sin(\theta)\sin(\varphi) & \rho\cos(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) & \rho\cos(\theta)\sin(\varphi) & \rho\sin(\theta)\cos(\varphi) \\ \cos(\varphi) & 0 & -\rho\sin(\varphi) \end{bmatrix}, \tag{3.7.20}$$

so, expanding along the third row,

$$\begin{aligned} \det\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} &= \cos(\varphi)(-\rho^2\sin^2(\theta)\sin(\varphi)\cos(\varphi) - \rho^2\cos^2(\theta)\sin(\varphi)\cos(\varphi)) \\ &\quad - \rho\sin(\varphi)(\rho\cos^2(\theta)\sin^2(\varphi) + \rho\sin^2(\theta)\sin^2(\varphi) \\ &= -\rho^2\sin(\varphi)\cos^2(\varphi)(\sin^2(\theta)+\cos^2(\theta)) - \rho^2\sin^3(\varphi)(\sin^2(\theta)+\cos^2(\theta)) \\ &= -\rho^2\sin(\varphi)\cos^2(\varphi) - \rho^2\sin^3(\varphi) \\ &= -\rho^2\sin(\varphi)(\cos^2(\varphi)+\sin^2(\varphi)) \\ &= -\rho^2\sin(\varphi). \end{aligned} \tag{3.7.21}$$

Now $\rho \geq 0$ and, since $0 \leq \varphi \leq \pi$, $\sin(\varphi) \geq 0$, so

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| = \rho^2\sin(\varphi). \tag{3.7.22}$$

Example In an earlier example we used the fact that the volume of a sphere of radius 1 is $\frac{4\pi}{3}$. In this example we will verify that the volume of a sphere of radius a is $\frac{4}{3}\pi a^3$. Let V be the volume of

$$D = \bar{B}^3((0,0,0),a),$$

the closed ball of radius a centered at the origin in $\mathbb{R}^3$. Then

$$V = \int\int\int_D dxdydz.$$

Although we may evaluate this integral using Cartesian coordinates, we will find it significantly easier to use spherical coordinates. Using the spherical coordinate change of variables

$$x = \rho\cos(\theta)\sin(\varphi),$$

$$y = \rho\sin(\theta)\sin(\varphi),$$

and

$$z = \rho\cos(\varphi),$$

the region D in xyz-space corresponds to the region

$$E = \{(\rho,\theta,\varphi) : 0 \leq \rho \leq a, 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi\}$$

in $\rho\theta\varphi$-space. Using (3.7.22) in the change of variables formula (3.7.5), we have

$$\begin{aligned}
V &= \int\int\int_D dxdydz \\
&= \int\int\int_E \left|\det\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| d\rho d\theta d\varphi \\
&= \int_0^1\int_0^{2\pi}\int_0^{\pi} \rho^2\sin(\varphi)d\varphi d\theta d\rho \\
&= \int_0^a\int_0^{2\pi} (-\rho^2\cos(\varphi))\Big|_0^{\pi} d\theta d\rho \\
&= \int_0^a\int_0^{2\pi} (-\rho^2(-1-1))d\theta d\rho \\
&= 2\int_0^a\int_0^{2\pi} \rho^2 d\theta d\rho \\
&= 4\pi\int_0^a \rho^2 d\rho \\
&= \frac{4\pi}{3}\rho^3\Big|_0^a \\
&= \frac{4}{3}\pi a^3.
\end{aligned}$$

Example Suppose we wish to evaluate

$$\int\int\int_D \log\sqrt{x^2+y^2+z^2}\; dxdydz,$$

where D is the region in $\mathbb{R}^3$ which lies between the two spheres with equations $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 4$ and above the xy-plane. Under the spherical coordinate change of variables

$$x = \rho\cos(\theta)\sin(\varphi),$$

$$y = \rho \sin(\theta)\sin(\varphi),$$

and

$$z = \rho \cos(\varphi),$$

the region D in xyz-space corresponds to the region

$$E = \left\{(\rho, \theta, \varphi) : 1 \leq \rho \leq 2, 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \frac{\pi}{2}\right\}$$

in $\rho\theta\varphi$-space. Using (3.7.22) in the change of variables formula (3.7.5), we have

$$\begin{aligned}
\int\!\!\int\!\!\int_D \log\sqrt{x^2+y^2+z^2}\; dxdydz &= \int\!\!\int\!\!\int_E \log(\rho)\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| d\rho d\theta d\varphi \\
&= \int_1^2\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \rho^2\log(\rho)\sin(\varphi)d\varphi d\theta d\rho \\
&= \int_1^2\int_0^{2\pi} (-\rho^2\log(\rho)\cos(\varphi))\Big|_0^{\frac{\pi}{2}} d\theta d\rho \\
&= \int_1^2\int_0^{2\pi} (-\rho^2\log(\rho))(0-1)d\theta d\rho \\
&= \int_1^2\int_0^{2\pi} \rho^2\log(\rho)d\theta d\rho \\
&= 2\pi\int_1^2 \rho^2\log(\rho)d\rho.
\end{aligned}$$

We use integration by parts to evaluate this final integral: letting

$$\begin{aligned}
u &= \log(\rho) & dv &= \rho^2 d\rho \\
du &= \frac{1}{\rho}\, d\rho & v &= \frac{\rho^3}{3},
\end{aligned}$$

we have

$$\begin{aligned}
\int\!\!\int\!\!\int_D \log\sqrt{x^2+y^2+z^2}\; dxdydz &= 2\pi\left(\frac{1}{3}\rho^3\log(\rho)\Big|_1^2 - \frac{1}{3}\int_1^2 \rho^2 d\rho\right) \\
&= \frac{16}{3}\pi\log(2) - \frac{2\pi\rho^3}{9}\Big|_1^2 \\
&= \frac{16}{3}\pi\log(2) - \frac{14\pi}{9} \\
&= \frac{2\pi}{3}\left(8\log(2) - \frac{7}{3}\right).
\end{aligned}$$

Problems

1. Find the area of the region enclosed by the ellipse with equation $x^2 + 4y^2 = 4$.

2. Given $a > 0$ and $b > 0$, show that the area enclosed by the ellipse with equation
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
is πab.

3. Find the volume of the region enclosed by the ellipsoid with equation
$$\frac{x^2}{25} + y^2 + \frac{z^2}{4} = 1.$$

4. Given $a > 0$, $b > 0$, and $c > 0$, show that the volume of the region enclosed by the ellipsoid
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$
is $\frac{4}{3}\pi abc$.

5. Find the polar coordinates for each of the following points given in Cartesian coordinates.
   (a) $(1, 1)$
   (b) $(-2, 3)$
   (c) $(-1, 3)$
   (d) $(4, -4)$

6. Find the Cartesian coordinates for each of the following points given in polar coordinates.
   (a) $(3, 0)$
   (b) $\left(2, \frac{5\pi}{6}\right)$
   (c) $(5, \pi)$
   (d) $\left(4, \frac{4\pi}{3}\right)$

7. Evaluate
$$\int\int_D (x^2 + y^2)dxdy,$$
where D is the disk in $\mathbb{R}^2$ of radius 2 centered at the origin.

8. Evaluate
$$\int\int_D \sin(x^2 + y^2)dxdy,$$
where D is the disk in $\mathbb{R}^2$ of radius 1 centered at the origin.

9. Evaluate
$$\int\int_D \frac{1}{x^2 + y^2}\, dxdy,$$
where D is the region in the first quadrant of $\mathbb{R}^2$ which lies between the circle with equation $x^2 + y^2 = 1$ and the circle with equation $x^2 + y^2 = 16$.

10. Evaluate
$$\int\int_D \log(x^2+y^2)dxdy,$$
where D is the region in $\mathbb{R}^2$ which lies between the circle with equation $x^2+y^2=1$ and the circle with equation $x^2+y^2=4$.

11. Using polar coordinates, verify that the area of a circle of radius r is πr^2.

12. Let
$$I=\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}dx.$$
    (a) Show that
    $$I^2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)}dxdy.$$
    (b) Show that
    $$I^2=\int_0^{\infty}\int_0^{2\pi} re^{-\frac{r^2}{2}}d\theta dr.$$
    (c) Show that
    $$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}dx=\sqrt{2\pi}.$$

13. Find the spherical coordinates of the point with Cartesian coordinates $(-1,1,2)$.

14. Find the spherical coordinates of the point with Cartesian coordinates $(3,2,-1)$.

15. Find the Cartesian coordinates of the point with spherical coordinates $\left(2,\frac{3\pi}{4},\frac{2\pi}{3}\right)$.

16. Find the Cartesian coordinates of the point with spherical coordinates $\left(5,\frac{5\pi}{3},\frac{\pi}{6}\right)$.

17. Evaluate
$$\int\int\int(x^2+y^2+z^2)dxdydz,$$
where D is the closed ball in $\mathbb{R}^3$ of radius 2 centered at the origin.

18. Evaluate
$$\int\int\int_D \frac{1}{\sqrt{x^2+y^2+z^2}}\,dxdydz,$$
where D is the region in $\mathbb{R}^3$ between the two spheres with equations $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=9$.

19. Evaluate
$$\int\int\int_D \sin(\sqrt{x^2+y^2+z^2}\,)dxdydz,$$
where D is the region in $\mathbb{R}^3$ described by $x\geq 0$, $y\geq 0$, $z\geq 0$, and $x^2+y^2+z^2\leq 1$.

20. Evaluate
$$\int\int\int_D e^{-(x^2+y^2+z^2)}dxdydz,$$
where D is the closed ball in $\mathbb{R}^3$ of radius 3 centered at the origin.

21. Let D be the region in $\mathbb{R}^3$ described by $x^2+y^2+z^2 \leq 1$ and $z \geq \sqrt{x^2+y^2}$.
    (a) Explain why the spherical coordinate change of variables maps the region
    $$E = \left\{(\rho, \theta, \varphi) : 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \frac{\pi}{4}\right\}$$
    onto D.
    (b) Find the volume of D.

22. If a point P has Cartesian coordinates (x, y, z), then the *cylindrical coordinates* of P are (r, θ, z), where r and θ are the polar coordinates of (x, y). Show that
$$\left|\det \frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r.$$

23. Use cylindrical coordinates to evaluate
$$\int\int_D \sqrt{x^2+y^2}dxdydz,$$
where D is the region in $\mathbb{R}^3$ described by $1 \leq x^2+y^2 \leq 4$ and $0 \leq z \leq 5$.

24. A drill with a bit with a radius of 1 centimeter is used to drill a hole through the center of a solid ball of radius 3 centimeters. What is the volume of the remaining solid?

25. Let D be the set of all points in the intersection of the two solid cylinders in $\mathbb{R}^3$ described by $x^2+y^2 \leq 1$ and $x^2+z^2 \leq 1$. Find the volume of D.