The Calculus of Functions

of

Several Variables

Section 3.6

Definite Integrals

We will first define the definite integral for a function $f : \mathbb{R}^2 \to \mathbb{R}$ and later indicate how the definition may be extended to functions of three or more variables.

Cartesian products

We will find the following notation useful. Given two sets of real numbers A and B, we define the *Cartesian product* of A and B to be the set

$$A \times B = \{(x, y) : x \in A, y \in B\}. \tag{3.6.1}$$

For example, if $A = \{1, 2, 3\}$ and $B = \{5, 6\}$, then

$$A \times B = \{(1, 5), (1, 6), (2, 5), (2, 6), (3, 5), (3, 6)\}.$$

In particular, if $a < b$, $c < d$, $A = [a, b]$, and $B = [c, d]$, then $A \times B = [a, b] \times [c, d]$ is the closed rectangle

$$\{(x, y) : a \leq x \leq b, c \leq y \leq d\},$$

as shown in Figure 3.6.1.

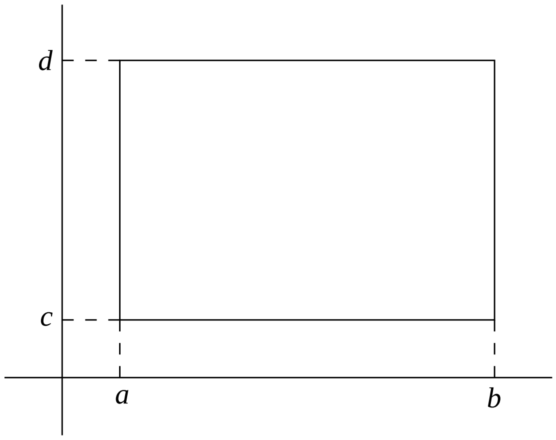

Figure 3.6.1 The closed rectangle $[a, b] \times [c, d]$

More generally, given real numbers $a_i < b_i$, $i = 1, 2, 3, \ldots, n$, we may write

$$[a_1, b_1] \times [a_2, b_2] \times \cdots [a_n, b_n]$$

for the closed rectangle

$$\{(x_1, x_2, \ldots, x_n) : a_i \leq x_i \leq b_i, i = 1, 2, \ldots, n\}$$

and

$$(a_1, b_1) \times (a_2, b_2) \times \cdots (a_n, b_n)$$

for the open rectangle

$$\{(x_1, x_2, \ldots, x_n) : a_i < x_i < b_i, i = 1, 2, \ldots, n\}.$$

Definite integrals on rectangles

Given $a < b$ and $c < d$, let

$$D = [a, b] \times [c, d]$$

and suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is defined on all of D. Moreover, we suppose f is *bounded* on D, that is, there exist constants m and M such that $m \leq f(x, y) \leq M$ for all (x, y) in D. In particular, the Extreme Value Theorem implies that f is bounded on D if f is continuous on D. Our definition of the definite integral of f over the rectangle D will follow the definition from one-variable calculus. Given positive integers m and n, we let P be a *partition* of $[a, b]$ into m intervals, that is, a set $P = \{x_0, x_1, \ldots, x_m\}$ where

$$a = x_0 < x_1 < \cdots < x_m = b, \tag{3.6.2}$$

and we let Q be a partition of $[c, d]$ into n intervals, that is, a set $Q = \{y_0, y_1, \ldots, y_n\}$ where

$$c = y_0 < y_1 < \cdots < y_n = d. \tag{3.6.3}$$

We will let $P \times Q$ denote the partition of D into mn rectangles

$$D_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j], \tag{3.6.4}$$

where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Note that D_{ij} has area $\Delta x_i \Delta y_j$, where

$$\Delta x_i = x_i - x_{i-1} \tag{3.6.5}$$

and

$$\Delta y_j = y_j - y_{j-1}. \tag{3.6.6}$$

An example is shown in Figure 3.6.2.

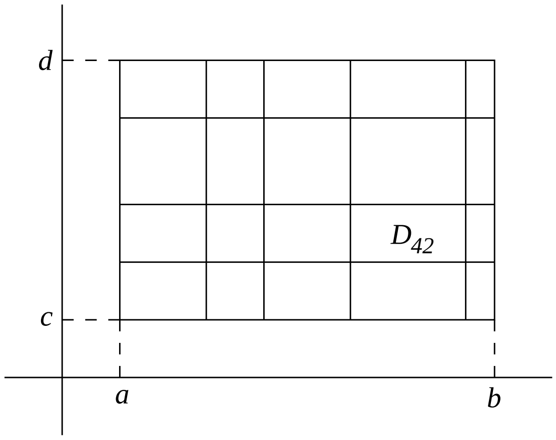

Figure 3.6.2 A partition of a rectangle $[a, b] \times [c, d]$

Now let m_{ij} be the largest real number with the property that $m_{ij} \leq f(x,y)$ for all (x,y) in D_{ij} and M_{ij} be the smallest real number with the property that $f(x,y) \leq M_{ij}$ for all (x,y) in D_{ij}. Note that if f is continuous on D, then m_{ij} is simply the minimum value of f on D_{ij} and M_{ij} is the maximum value of f on D_{ij}, both of which are guaranteed to exist by the Extreme Value Theorem. If f is not continuous, our assumption that f is bounded nevertheless guarantees the existence of the m_{ij} and M_{ij}, although the justification for this statement lies beyond the scope of this book.

We may now define the *lower sum*, $L(f, P \times Q)$, for f with respect to the partition $P \times Q$ by

$$L(f, P \times Q) = \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} \Delta x_i \Delta y_j \tag{3.6.7}$$

and the *upper sum*, $U(f, P \times Q)$, for f with respect to the partition $P \times Q$ by

$$U(f, P \times Q) = \sum_{i=1}^{m} \sum_{j=1}^{n} M_{ij} \Delta x_i \Delta y_j. \tag{3.6.8}$$

Geometrically, if $f(x,y) \geq 0$ for all (x,y) in D and V is the volume of the region which lies beneath the graph of f and above the rectangle D, then $L(f, P \times Q)$ and $U(f, P \times Q)$ represent lower and upper bounds, respectively, for V. (See Figure 3.6.3 for an example of one term of a lower sum). Moreover, we should expect that these bounds can be made arbitrarily close to V using sufficiently fine partitions P and Q. In part this implies that we may characterize V as the only real number which lies between $L(f, P \times Q)$ and $U(f, P \times Q)$ for all choices of partitions P and Q. This is the basis for the following definition.

Definition Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is bounded on the rectangle $D = [a,b] \times [c,d]$. With the notation as above, we say f is *integrable* on D if there exists a unique real number I such that

$$L(f, P \times Q) \leq I \leq U(f, P \times Q) \tag{3.6.9}$$

for all partitions P of $[a,b]$ and Q of $[c,d]$. If f is integrable on D, we call I the *definite integral* of f on D, which we denote

$$I = \int\int_D f(x,y)\,dx\,dy. \tag{3.6.10}$$

Geometrically, if $f(x,y) \geq 0$ for all (x,y) in D, we may think of the definite integral of f on D as the volume of the region in R^3 which lies beneath the graph of f and above the rectangle D. Other interpretations include total mass of the rectangle D (if $f(x,y)$ represents the density of mass at the point (x,y)) and total electric charge of the rectangle D (if $f(x,y)$ represents the charge density at the point (x,y)).

Example Suppose $f(x,y) = x^2 + y^2$ and $D = [0,1] \times [0,3]$. If we let

$$P = \{0, \frac{1}{2}, 1\}$$

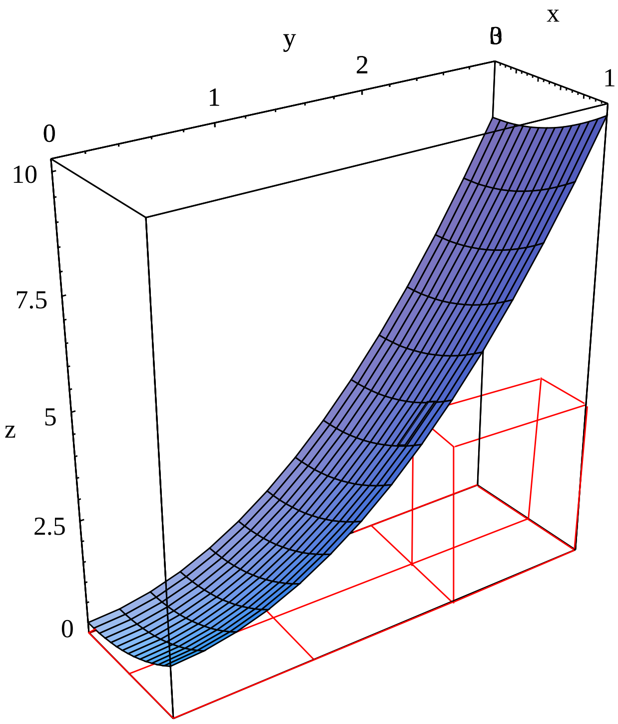

Figure 3.6.3 Graph of $f(x, y) = x^2 + y^2$ showing one term of a lower sum

and

$$Q = \{0, 1, 2, 3\},$$

then the minimum value of f on each rectangle of the partition $P \times Q$ occurs at the lower left-hand corner of the rectangle and the maximum value of f occurs at the upper right-hand corner of the rectangle. See Figure 3.6.3 for a picture of one term of the lower sum. Hence

$$\begin{aligned} L(f, P \times Q) &= f(0,0) \times \frac{1}{2} \times 1 + f\left(\frac{1}{2}, 0\right) \times \frac{1}{2} \times 1 + f(0,1) \times \frac{1}{2} \times 1 \\ &\quad + f\left(\frac{1}{2}, 1\right) \times \frac{1}{2} \times 1 + f(0,2) \times \frac{1}{2} \times 1 + f\left(\frac{1}{2}, 2\right) \times \frac{1}{2} \times 1 \\ &= 0 + \frac{1}{8} + \frac{1}{2} + \frac{5}{8} + 2 + \frac{17}{8} \\ &= \frac{43}{8} = 5.375 \end{aligned}$$

and

$$\begin{aligned} U(f, P \times Q) &= f\left(\frac{1}{2}, 1\right) \times \frac{1}{2} \times 1 + f(1,1) \times \frac{1}{2} \times 1 + f\left(\frac{1}{2}, 2\right) \times \frac{1}{2} \times 1 \\ &\quad + f(1,2) \times \frac{1}{2} \times 1 + f\left(\frac{1}{2}, 3\right) \times \frac{1}{2} \times 1 + f(1,3) \times \frac{1}{2} \times 1 \end{aligned}$$

$$= \frac{5}{8} + 1 + \frac{17}{8} + \frac{5}{2} + \frac{37}{8} + 5$$
$$= \frac{127}{8} = 15.875.$$

We will see below that the continuity of f implies that f is integrable on D, so we may conclude that

$$5.375 \leq \int\int (x^2 + y^2)dxdy \leq 15.875.$$

Example Suppose k is a constant and $f(x, y) = k$ for all (x, y) in the rectangle $D = [a, b] \times [c, d]$. The for any partitions $P = \{x_0, x_1, \ldots, x_m\}$ of $[a, b]$ and $Q = \{y_0, y_1, \ldots, y_n\}$ of $[c, d]$, $m_{ij} = k = M_{ij}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Hence

$$\begin{aligned} L(f, P \times Q) &= U(f, P \times Q) \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} k \Delta x_i \Delta y_j \\ &= k \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta x_i \Delta y_j \\ &= k \times (\text{area of } D) \\ &= k(b-a)(d-c). \end{aligned}$$

Hence f is integrable and

$$\int\int_D f(x, y)dxdy = \int\int_D kdxdy = k(b-a)(d-c).$$

Of course, geometrically this result is saying that the volume of a box with height k and base D is k times the area of D. In particular, if $k = 1$ we see that

$$\int\int_D dxdy = \text{area of } D.$$

Example If $D = [1, 2] \times [-1, 3]$, then

$$\int_D 5dxdy = 5(2-1)(3+1) = 20.$$

The properties of the definite integral stated in the following proposition follow easily from the definition, although we will omit the somewhat technical details.

Proposition Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are both integrable on the rectangle $D = [a, b] \times [c, d]$ and k is a scalar constant. Then

$$\int\int_D (f(x, y) + g(x, y))dxdy = \int\int_D f(x, y)dxdy + \int\int_D g(x, y)dxdy, \tag{3.6.11}$$

$$\int\int_D kf(x,y)dxdy = k\int\int_D f(x,y)dxdy, \tag{3.6.12}$$

and, if $f(x,y) \le g(x,y)$ for all (x,y) in D,

$$\int\int_D f(x,y)dxdy \le \int\int_D g(x,y)dxdy. \tag{3.6.13}$$

Our definition does not provide a practical method for determining whether a given function is integrable or not. A complete characterization of integrability is beyond the scope of this text, but we shall find one simple condition very useful: if f is continuous on an open set containing the rectangle D, then f is integrable on D. Although we will not attempt a full proof of this result, the outline is as follows. If f is continuous on $D = [a,b] \times [c,d]$ and we are given any $\epsilon > 0$, then it is possible to find partitions P of $[a,b]$ and Q of $[c,d]$ sufficiently fine to guarantee that if (x,y) and (u,v) are points in the same rectangle D_{ij} of the partition $P \times Q$ of D, then

$$|f(x,y) - f(u,v)| < \frac{\epsilon}{(b-a)(d-c)}. \tag{3.6.14}$$

(Note that this is not a direct consequence of the continuity of f, but follows from a slightly deeper property of continuous functions on closed bounded sets known as *uniform continuity*.) It follows that if m_{ij} is the minimum value and M_{ij} is the maximum value of f on D_{ij}, then

$$\begin{aligned} U(f,P\times Q) - L(f,P\times Q) &= \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\Delta x_i \Delta y_j - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\Delta x_i \Delta y_j \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n}(M_{ij} - m_{ij})\Delta x_i \Delta y_j \\ &< \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\epsilon}{(b-a)(d-c)}\Delta x_i \Delta y_j \\ &= \frac{\epsilon}{(b-a)(d-c)}\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta x_i \Delta y_j \\ &= \frac{\epsilon}{(b-a)(d-c)}(b-a)(d-c) \\ &= \epsilon. \end{aligned} \tag{3.6.15}$$

It now follows that we may find upper and lower sums which are arbitrarily close, from which follows the integrability of f.

Theorem If f is continuous on an open set containing the rectangle D, then f is integrable on D.

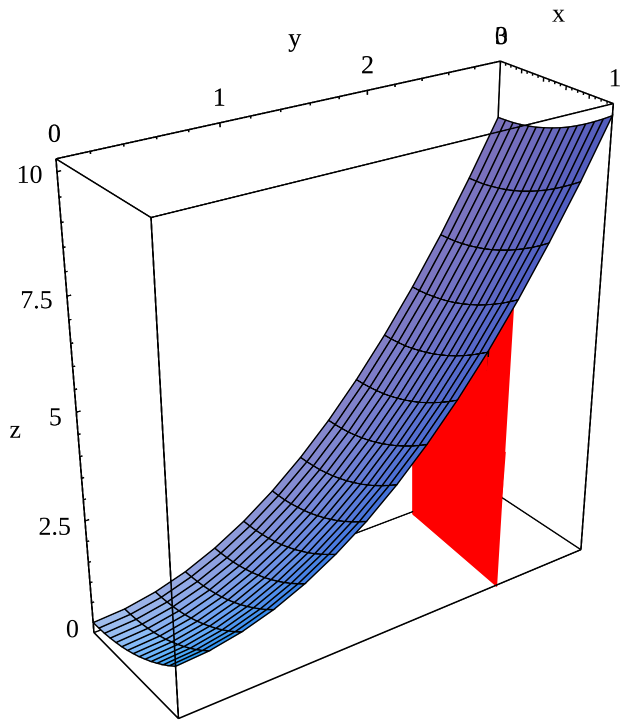

Figure 3.6.4 A slice of the region beneath $f(x, y) = x^2 + y^2$ with area $\alpha(2)$

Example If $f(x, y) = x^2 + y^2$, then f is continuous on all of R^2. Hence f is integrable on $D = [0, 1] \times [0, 3]$.

Iterated integrals

Now suppose we have a rectangle $D = [a, b] \times [c, d]$ and a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y) \geq 0$ for all (x, y) in D. Let

$$B = \{(x, y, z) : (x, y) \in D, 0 \leq z \leq f(x, y)\}. \tag{3.6.16}$$

Then B is the region in $\mathbb{R}^3$ bounded below by D and above by the graph of f. If we let V be the volume of B, then

$$V = \int\int_D f(x, y)dxdy. \tag{3.6.17}$$

However, there is another approach to finding V. If, for every $c \leq y \leq d$, we let

$$\alpha(y) = \int_a^b f(x, y)dx, \tag{3.6.18}$$

then $\alpha(y)$ is the area of a slice of B cut by a plane orthogonal to both the xy-plane and the yz-plane and passing through the point $(0, y, 0)$ on the y-axis (see Figure 3.6.4 for an example). If we let the partition $Q = \{y_0, y_1, \ldots, y_n\}$ divide $[c, d]$ into n intervals of equal length Δy, then we may approximate V by

$$\sum_{j=1}^{n} \alpha(y_i)\Delta y. \tag{3.6.19}$$

That is, we may approximate V by slicing B into slabs of thickness Δy perpendicular to the yz-plane, and then summing approximations to the volume of each slab. As n increases, this approximation should converge to V; at the same time, since (3.6.19) is a right-hand rule approximation to the definite integral of α over $[c,d]$, the sum should converge to

$$\int_c^d \alpha(y)dy$$

as n increases. That is, we should have

$$V = \lim_{n\to\infty} \sum_{j=1}^{n} \alpha(y_i)\Delta y = \int_c^d \alpha(y)dy = \int_c^d \left(\int_a^b f(x,y)dx\right) dy. \tag{3.6.20}$$

Note that the expression on the right-hand side of (3.6.20) is not the definite integral of f over D, but rather two successive integrals of one variable. Also, we could have reversed our order and first integrated with respect to y and then integrated the result with respect to x.

Definition Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is defined on a rectangle $D = [a,b] \times [c,d]$. The *iterated integrals* of f over D are

$$\int_c^d \int_a^b f(x,y)dxdy = \int_c^d \left(\int_a^b f(x,y)dx\right) dy \tag{3.6.21}$$

and

$$\int_a^b \int_c^d f(x,y)dydx = \int_a^b \left(\int_c^d f(x,y)dy\right) dx. \tag{3.6.22}$$

In the situation of the preceding paragraph, we should expect the iterated integrals in (3.6.21) and (3.6.22) to be equal since they should both equal V, the volume of the region B. Moreover, since we also know that

$$V = \int\int_D f(x,y)dxdy,$$

the iterated integrals should both be equal to the definite integral of f over D. These statements may in fact be verified as long as f is integrable on D and the iterated integrals exist. In this case, iterated integrals provide a method of evaluating double integrals in terms of integrals of a single variable (for which we may use the Fundamental Theorem of Calculus).

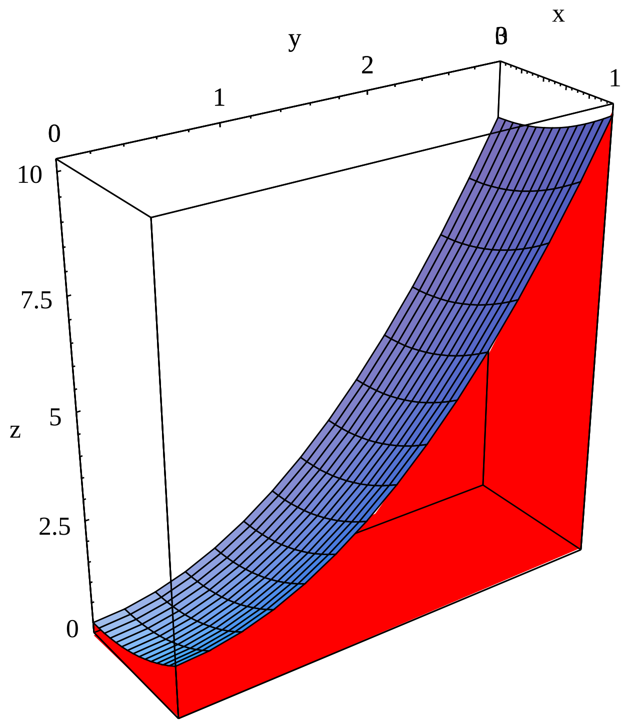

Figure 3.6.5 Region beneath $f(x,y) = x^2 + y^2$ over the rectangle $[0,1] \times [0,3]$

Fubini's Theorem (for rectangles) Suppose f is integrable over the rectangle $D = [a,b] \times [c,d]$. If

$$\int_c^d \int_a^b f(x,y)dxdy$$

exists, then

$$\int\int_D f(x,y)dxdy = \int_c^d \int_a^b f(x,y)dxdy. \tag{3.6.23}$$

If

$$\int_a^b \int_c^d f(x,y)dydx$$

exists, then

$$\int\int_D f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dxdy. \tag{3.6.24}$$

Example To find the volume V of the region beneath the graph of $f(x,y) = x^2 + y^2$ and over the rectangle $D = [0,1] \times [0,3]$ (as shown in Figure 3.6.5), we compute

$$\begin{aligned} V &= \int\int_D (x^2 + y^2)dxdy \\ &= \int_0^3 \int_0^1 (x^2 + y^2)dxdy \end{aligned}$$

$$
\begin{aligned}
&= \int_0^3 \left(\frac{x^3}{3} + xy^2\right)\Big|_0^1 dy \\
&= \int_0^3 \left(\frac{1}{3} + y^2\right) dy \\
&= \left(\frac{y}{3} + \frac{y^3}{3}\right)\Big|_0^3 \\
&= 1 + 9 \\
&= 10.
\end{aligned}
$$

We could also compute the iterated integral in the other order:

$$
\begin{aligned}
V &= \int\int_D (x^2 + y^2)dxdy \\
&= \int_0^1 \int_0^3 (x^2 + y^2)dydx \\
&= \int_0^1 \left(x^2 y + \frac{y^3}{3}\right)\Big|_0^3 dx \\
&= \int_0^1 (3x^2 + 9)dx \\
&= (x^3 + 9y)\big|_0^1 \\
&= 1 + 9 \\
&= 10.
\end{aligned}
$$

Example If $D = [1,2] \times [0,1]$, then

$$\int\int_D x^2 y dxdy = \int_1^2 \int_0^1 x^2 y dydx = \int_1^2 \frac{x^2 y^2}{2}\Big|_0^1 dx = \int_1^2 \frac{x^2}{2}\, dx = \frac{x^3}{6}\Big|_1^2 = \frac{8}{6} - \frac{1}{6} = \frac{7}{6}.$$

Definite integrals on other regions

Integrals over intervals suffice for most applications of functions of a single variable. However, for functions of two variables it is important to consider integrals on regions other than rectangles. To extend our definition, consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ defined on a bounded region D. Let D^* be a rectangle containing D and, for any (x,y) in D^*, define

$$f^*(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in D, \\ 0, & \text{if } (x,y) \notin D. \end{cases} \tag{3.6.25}$$

In other words, f^* is identical to f on D and 0 at all points of D^* outside of D. Now if f^* is integrable on D^*, and since the the region where f^* is 0 should contribute nothing

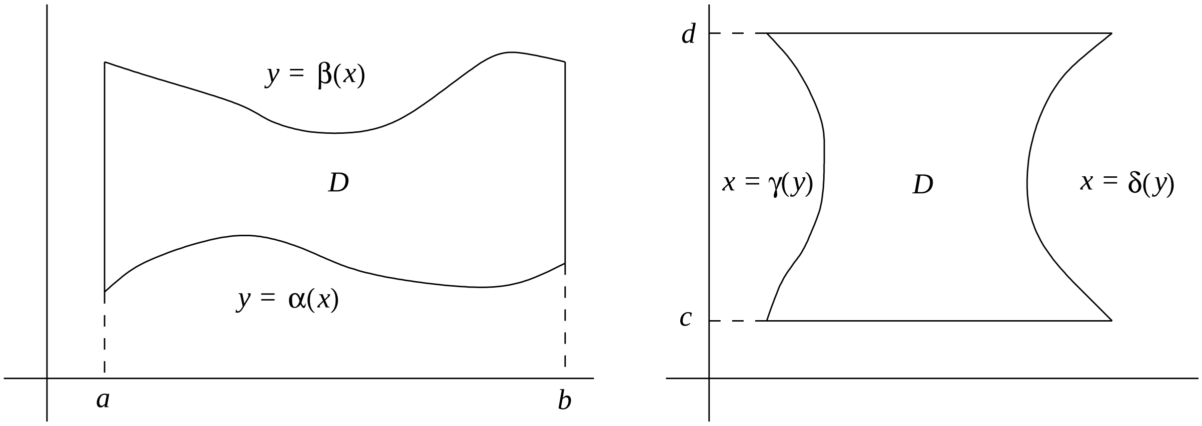

Figure 3.6.6 Regions of Type I and Type II

to the value of the integral, it is reasonable to define the integral of f over D to be equal to the integral of f^* over D^*.

Definition Suppose f is defined on a bounded region D of $\mathbb{R}^2$ and let D^* be any rectangle containing D. Define f^* as in (3.6.25). We say f is integrable on D if f^* is integrable on D^*, in which case we define

$$\int\int_D f(x,y)dxdy = \int\int_{D^*} f^*(x,y)dxdy. \tag{3.6.26}$$

Note that the integrability of f on a region D depends not only on the nature of f, but on the region D as well. In particular, even if f is continuous on an open set containing D, it may still turn out that f is not integrable on D because of the complicated nature of the boundary of D. Fortunately, there are two basic types of regions which occur frequently and to which our previous theorems generalize.

Definition We say a region D in $\mathbb{R}^2$ is of *Type I* if there exist real numbers $a < b$ and continuous functions $\alpha : \mathbb{R} \to \mathbb{R}$ and $\beta : \mathbb{R} \to \mathbb{R}$ such that $\alpha(x) \leq \beta(x)$ for all x in $[a,b]$ and

$$D = \{(x,y) : a \leq x \leq b, \alpha(x) \leq y \leq \beta(x)\}. \tag{3.6.27}$$

We say a region D in R^2 is of *Type II* if there exist real numbers $c < d$ and continuous functions $\gamma : \mathbb{R} \to \mathbb{R}$ and $\delta : \mathbb{R} \to \mathbb{R}$ such that $\gamma(y) \leq \delta(y)$ for all y in $[c,d]$ and

$$D = \{(x,y) : c \leq y \leq d, \gamma(y) \leq x \leq \delta(y)\}. \tag{3.6.28}$$

Figure 3.6.6 shows typical examples of regions of Type I and Type II.

Example If D is the triangle with vertices at $(0,0)$, $(1,0)$, and $(1,1)$, then

$$D = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq x\}.$$

Hence D is a Type I region with $\alpha(x) = 0$ and $\beta(x) = x$. Note that we also have

$$D = \{(x,y) : 0 \leq y \leq 1, y \leq x \leq 1\},$$

so D is also a Type II region with $\gamma(y) = y$ and $\delta(y) = 1$. See Figure 3.6.7.

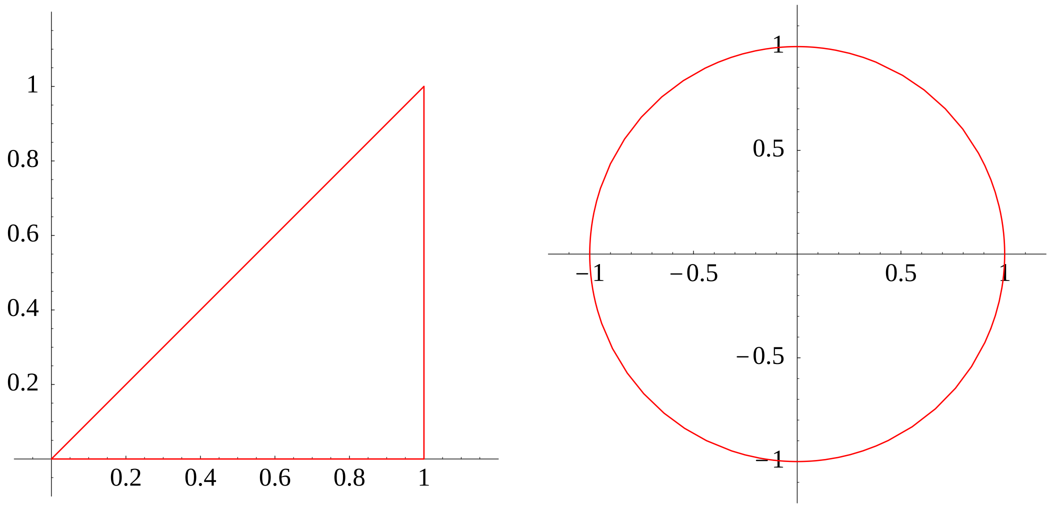

Figure 3.6.7 Two regions which are of both Type I and Type II

Example The closed disk

$$D = \{(x, y) : x^2 + y^2 \leq 1\}$$

is both a region of Type I, with

$$D = \{(x, y) : -1 \leq x \leq 1, -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}\},$$

and a region of Type II, with

$$D = \{(x, y) : -1 \leq y \leq 1, -\sqrt{1 - y^2} \leq x \leq \sqrt{1 - x^2}\}.$$

See Figure 3.6.7.

Example Let D be the region which lies beneath the graph of $y = x^2$ and above the interval $[-1, 1]$ on the x-axis. Then

$$D = \{(x, y) : -1 \leq x \leq 1, 0 \leq y \leq x^2\},$$

so D is a region of Type I. However, D is not a region of Type II. See Figure 3.6.8.

Theorem If D is a region of Type I or a region of Type II and $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous on an open set containing D, then f is integrable on D.

Fubini's Theorem (for regions of Type I and Type II) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is integrable on the region D. If D is a region of Type I with

$$D = \{(x, y) : a \leq x \leq b, \alpha(x) \leq y \leq \beta(x)\}$$

and the iterated integral

$$\int_a^b \int_{\alpha(x)}^{\beta(x)} f(x, y)dydx$$

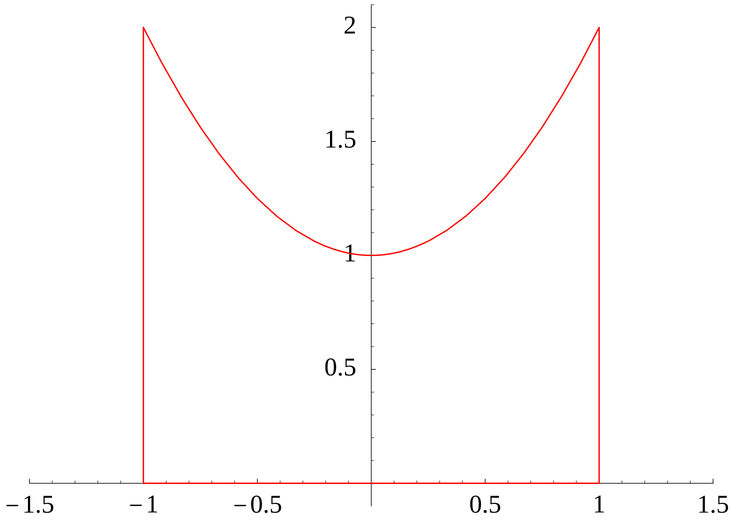

Figure 3.6.8 A region which is of Type I but not of Type II

exists, then

$$\int\int_D f(x,y)dxdy = \int_a^b \int_{\alpha(x)}^{\beta(x)} f(x,y)dydx. \tag{3.6.29}$$

If D is a region of Type II with

$$D = \{(x,y) : c \le y \le d, \gamma(y) \le x \le \delta(y)\}$$

and the iterated integral

$$\int_c^d \int_{\gamma(y)}^{\delta(y)} f(x,y)dxdy$$

exists, then

$$\int\int_D f(x,y)dxdy = \int_c^d \int_{\gamma(y)}^{\delta(y)} f(x,y)dydx. \tag{3.6.30}$$

Example Let D be the triangle with vertices at $(0,0)$, $(1,0)$, and $(1,1)$, as in the example above. Expressing D as a region of Type I, we have

$$\int\int_D xydxdy = \int_0^1 \int_0^x xydydx = \int_0^1 \left.\frac{xy^2}{2}\right|_0^x dx = \int_0^1 \frac{x^3}{2}\,dx = \left.\frac{x^4}{8}\right|_0^1 = \frac{1}{8}.$$

Since D is also a region of Type II, we may evaluate the integral in the other order as well, obtaining

$$\int\int_D xydxdy = \int_0^1 \int_y^1 xydxdy = \int_0^1 \left.\frac{x^2y}{2}\right|_y^1 dy = \int_0^1 \left(\frac{y}{2} - \frac{y^3}{2}\right) dy = \left.\left(\frac{y^2}{4} - \frac{y^4}{8}\right)\right|_0^1 = \frac{1}{8}.$$

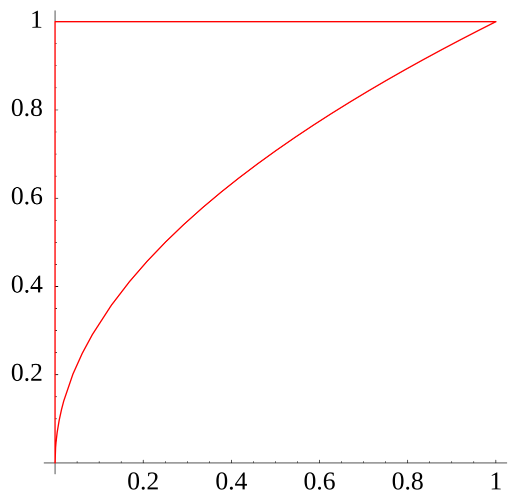

Figure 3.6.9 The region $D = \{(x, y) : 0 \leq x \leq 1, \sqrt{x} \leq y \leq 1\}$

In the last example the choice of integration was not too important, with the first order being perhaps slightly easier than the second. However, there are times when the choice of the order of integration has a significant effect on the ease of integration.

Example Let $D = \{(x, y) : 0 \leq x \leq 1, \sqrt{x} \leq y \leq 1\}$ (see Figure 3.6.9). Since D is both of Type I and of Type II, we may evaluate

$$\int\int_D e^{-y^3}\,dx dy$$

either as

$$\int_0^1 \int_{\sqrt{x}}^1 e^{-y^3}\,dy dx$$

or as

$$\int_0^1 \int_0^{y^2} e^{-y^3}\,dx dy.$$

The first of these two iterated integrals requires integrating $g(y) = e^{-y^3}$; however, we may evaluate the second easily:

$$\begin{aligned}\int\int_D e^{-y^3}\,dx dy &= \int_0^1 \int_0^{y^2} e^{-y^3}\,dx dy \\ &= \int_0^1 x e^{-y^3}\Big|_0^{y^2} dy \\ &= \int_0^1 y^2 e^{-y^3}\,dy \\ &= -\frac{1}{3}e^{-y^3}\Big|_0^1 \\ &= \frac{1}{3}(1 - e^{-1}).\end{aligned}$$

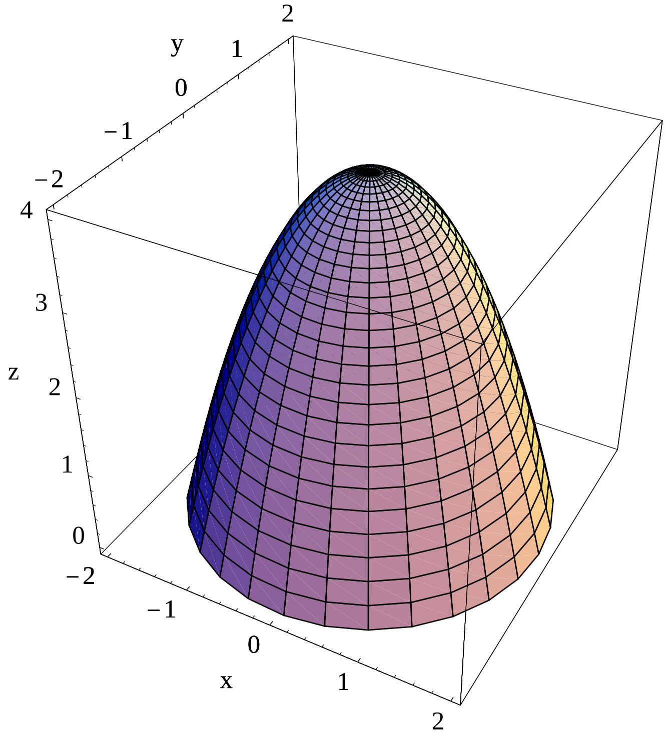

Figure 3.6.10 Region bounded by $z = 4 - x^2 - y^2$ and the xy-plane

Example Let V be the volume of the region lying below the paraboloid P with equation $z = 4 - x^2 - y^2$ and above the xy-plane (see Figure 3.6.10). Since the surface P intersects the xy-plane when

$$4 - x^2 - y^2 = 0,$$

that is, when

$$x^2 + y^2 = 4,$$

V is the volume of the region bounded above by the graph of $f(x, y) = 4 - x^2 - y^2$ and below by the region

$$D = \{(x, y) : x^2 + y^2 \leq 4\}.$$

If we describe D as a Type I region, namely,

$$D = \{(x, y) : -2 \leq x \leq 2, -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}\},$$

then we may compute

$$\begin{aligned} V &= \int\int_D (4 - x^2 - y^2)dxdy \\ &= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - y^2)dydx \\ &= \int_{-2}^{2} \left(4y - x^2 y - \frac{y^3}{3}\right)\Bigg|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx \end{aligned}$$

$$= \int_{-2}^{2} \left(8\sqrt{4-x^2} - 2x^2\sqrt{4-x^2} - \frac{2}{3}(4-x^2)^{\frac{3}{2}}\right) dx$$
$$= 2\int_{-2}^{2} \left((4-x^2)\sqrt{4-x^2} - \frac{1}{3}(4-x^2)^{\frac{3}{2}}\right) dx$$
$$= \frac{4}{3}\int_{-2}^{2} (4-x^2)^{\frac{3}{2}} dx.$$

Using the substitution $x = 2\sin(\theta)$, we have $dx = 2\cos(\theta)d\theta$, and so

$$\begin{aligned}
V &= \frac{4}{3}\int_{-2}^{2} (4-x^2)^{\frac{3}{2}} dx \\
&= \frac{4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4-4\sin^2(\theta))^{\frac{3}{2}} 2\cos(\theta)d\theta \\
&= \frac{64}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4(\theta)d\theta \\
&= \frac{64}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\cos(2\theta)}{2}\right)^2 d\theta \\
&= \frac{16}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+2\cos(2\theta)+\cos^2(2\theta))d\theta \\
&= \frac{16}{3}\left(\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \sin(2\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos(4\theta)}{2} d\theta\right) \\
&= \frac{16}{3}\left(\pi + \frac{\theta}{2}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{8}\sin(4\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right) \\
&= \frac{16}{3}\left(\pi + \frac{\pi}{2}\right) \\
&= 8\pi.
\end{aligned}$$

Integrals of functions of three or more variables

We will now sketch how to extend the definition of the definite integral to higher dimensions. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is bounded on an n-dimensional closed rectangle

$$D = [a_1, b_1] \times [a_2, b_2] \times \cdots [a_n, b_n].$$

Let P_1, P_2, ..., P_n partition the intervals $[a_1, b_1]$, $[a_2, b_2]$, ..., $[a_n, b_n]$ into m_1, m_2, ..., m_n, respectively, intervals, and let $P_1 \times P_2 \times \cdots \times P_n$ represent the corresponding partition of D into $m_1 m_2 \cdots m_n$ n-dimensional closed rectangles $D_{i_1 i_2 \cdots i_n}$. If $m_{i_1 i_2 \cdots i_n}$ is the largest real number such that $m_{i_1 i_2 \cdots i_n} \leq f(\mathbf{x})$ for all $\mathbf{x}$ in $D_{i_1 i_2 \cdots i_n}$ and $M_{i_1 i_2 \cdots i_n}$ is the smallest real number such that $f(\mathbf{x}) \leq M_{i_1 i_2 \cdots i_n}$ for all $\mathbf{x}$ in $D_{i_1 i_2 \cdots i_n}$, then we may define the lower sum

$$L(f, P_1 \times P_2 \times \cdots \times P_n) = \sum_{i_1=1}^{m_1} \sum_{i_2}^{m_2} \cdots \sum_{i_n=1}^{m_n} m_{i_1 i_2 \cdots i_n} \Delta x_{1 i_1} \Delta x_{2 i_2} \cdots \Delta x_{n i_n} \tag{3.6.31}$$

and the upper sum

$$U(f, P_1 \times P_2 \times \cdots \times P_n) = \sum_{i_1=1}^{m_1} \sum_{i_2}^{m_2} \cdots \sum_{i_n=1}^{m_n} M_{i_1 i_2 \cdots i_n} \Delta x_{1 i_1} \Delta x_{2 i_2} \cdots \Delta x_{n i_n}, \tag{3.6.32}$$

where Δx_{jk} is the length of the kth interval of the partition P_j. We then say f is *integrable* on D if there exists a unique real number I with the property that

$$L(f, P_1 \times P_2 \times \cdots \times P_n) \leq I \leq U(f, P_1 \times P_2 \times \cdots \times P_n) \tag{3.6.33}$$

for all choices of partitions P_1, P_2, $\ldots$, P_n and we write

$$I = \int \cdots \int \int_D f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \cdots dx_n, \tag{3.6.34}$$

or

$$I = \int \cdots \int \int_D f(\mathbf{x}) d\mathbf{x}, \tag{3.6.35}$$

for the *definite integral* of f on D.

We may now generalize the definition of the integral to more general regions in the same manner as above. Moreover, our integrability theorem and Fubini's theorem, with appropriate changes, hold as well. When $n = 3$, we may interpret

$$\int \int \int_D f(x, y, z) dx dy dz \tag{3.6.36}$$

to be the total mass of D if $f(x, y, z)$ represents the density of mass at (x, y, z), or the total electric charge of D if $f(x, y, z)$ represents the electric charge density at (x, y, z). For any value of n we may interpret

$$\int \cdots \int \int_D dx_1 dx_2 \cdots dx_n \tag{3.6.37}$$

to be the n-dimensional volume of D. We will not go into further details, preferring to illustrate with examples.

Example Suppose D is the closed rectangle

$$\begin{aligned} D &= \{(x, y, z, t) : 0 \leq x \leq 1, -1 \leq y \leq 1, -2 \leq z \leq 2, 0 \leq t \leq 2\} \\ &= [0, 1] \times [-1, 1] \times [-2, 2] \times [0, 2]. \end{aligned}$$

Then

$$\begin{aligned}
\int\int\int\int_D (x^2+y^2+z^2-t^2)dxdydzdt &= \int_0^1\int_{-1}^1\int_{-2}^2\int_0^2 (x^2+y^2+z^2-t^2)dtdzdydx \\
&= \int_0^1\int_{-1}^1\int_{-2}^2 \left(x^2t+y^2t+z^2t-\frac{t^3}{3}\right)\Big|_0^2 dzdydx \\
&= \int_0^1\int_{-1}^1\int_{-2}^2 \left(2x^2+2y^2+2z^2-\frac{8}{3}\right) dzdxdy \\
&= \int_0^1\int_{-1}^1 \left(2x^2z+2y^2z+\frac{2z^3}{3}-\frac{8z}{3}\right)\Big|_{-2}^2 dydx \\
&= \int_0^1\int_{-1}^1 \left(8x^2+8y^2+\frac{32}{3}-\frac{32}{3}\right) dydx \\
&= \int_0^1 \left(8x^2y+\frac{8y^2}{3}\right)\Big|_{-1}^1 dx \\
&= \int_0^1 \left(16x^2+\frac{16}{3}\right) dx \\
&= \left(\frac{16x^3}{3}+\frac{16x}{3}\right)\Big|_0^1 \\
&= \frac{32}{3}.
\end{aligned}$$

Example Let D be the region in $\mathbb{R}^3$ bounded by the the three coordinate planes and the plane P with equation $z = 1-x-y$ (see Figure 3.6.11). Suppose we wish to evaluate

$$\int\int\int_D xyzdxdydz.$$

Note that the side of D which lies in the xy-plane, that is, the plane $z=0$, is a triangle with vertices at $(0,0,0)$, $(1,0,0)$, and $(0,1,0)$. Or, strictly in terms of x and y coordinates, we may describe this face as the triangle in the first quadrant bounded by the line $y=1-x$ (see Figure 3.6.11). Hence x varies from 0 to 1, and, for each value of x, y varies from 0 to $1-x$. Finally, once we have fixed a values for x and y, z varies from 0 up to P, that is, to $1-x-y$. Hence we have

$$\begin{aligned}
\int\int\int_D xyzdxdydz &= \int_0^1\int_0^{1-x}\int_0^{1-x-y} xyzdzdydx \\
&= \int_0^1\int_0^{1-x} \frac{xyz^2}{2}\Big|_0^{1-x-y} dydx \\
&= \int_0^1\int_0^{1-x} \frac{xy(1-x-y)^2}{2} dydx
\end{aligned}$$

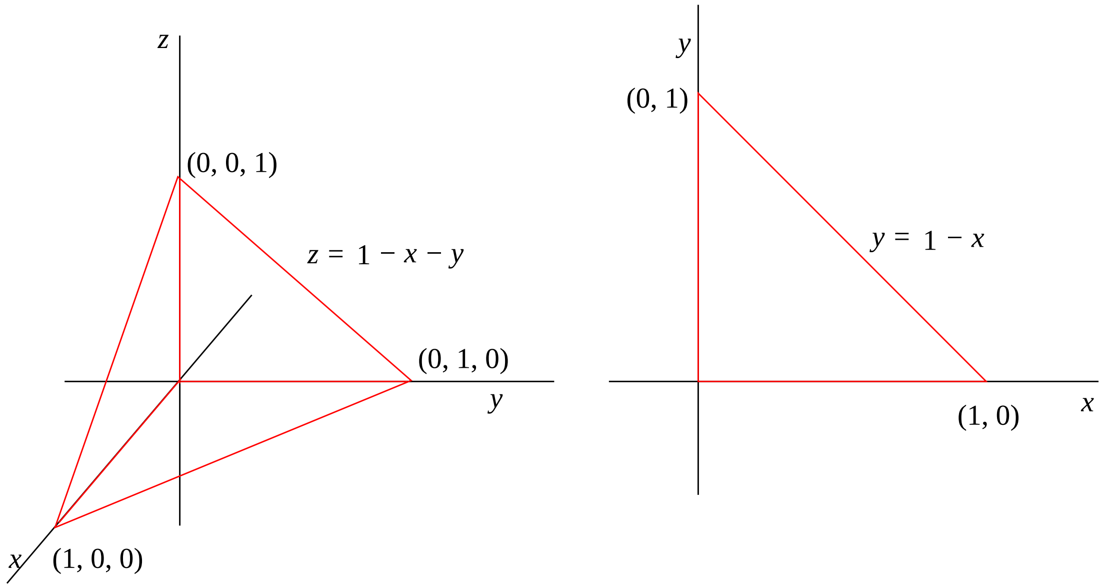

Figure 3.6.11 Region bounded by the coordinate planes and the plane $z = 1 - x - y$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^1 \int_0^{1-x} (xy - 2x^2y + x^3y + 2x^2y^2 + xy^3)dydx \\
&= \frac{1}{2}\int_0^1 \left(\frac{xy^2}{2} - 2x^2y^2 + \frac{x^3y^2}{2} + \frac{2x^2y^3}{3} + \frac{xy^4}{4}\right)\Big|_0^{1-x} dx \\
&= \frac{1}{2}\int_0^1 \left(\frac{3x}{4} - \frac{10x^2}{3} + \frac{9x^3}{2} - 2x^4 + \frac{x^5}{12}\right) dx \\
&= \frac{1}{2}\int_0^1 \left(\frac{3x^2}{8} - \frac{10x^3}{9} + \frac{9x^4}{8} - \frac{2x^5}{5} + \frac{x^6}{72}\right)\Big|_0^1 \\
&= \frac{1}{2}\left(\frac{3}{8} - \frac{10}{9} + \frac{9}{8} - \frac{2}{5} + \frac{1}{72}\right) \\
&= \frac{1}{720}.
\end{aligned}
$$

Example Let V be the volume of the region D in $\mathbb{R}^3$ bounded by the paraboloids with equations $z = 10 - x^2 - y^2$ and $z = x^2 + y^2 - 8$ (see Figure 3.6.12). We will find V by evaluating

$$V = \int\int\int_D dxdydz.$$

To set up an iterated integral, we first note that the paraboloid $z = 10 - x^2 - y^2$ opens downward about the z-axis and the paraboloid $z = x^2 + y^2 - 8$ opens upward about the z axis. The two paraboloids intersect when

$$10 - x^2 - y^2 = x^2 + y^2 - 8,$$

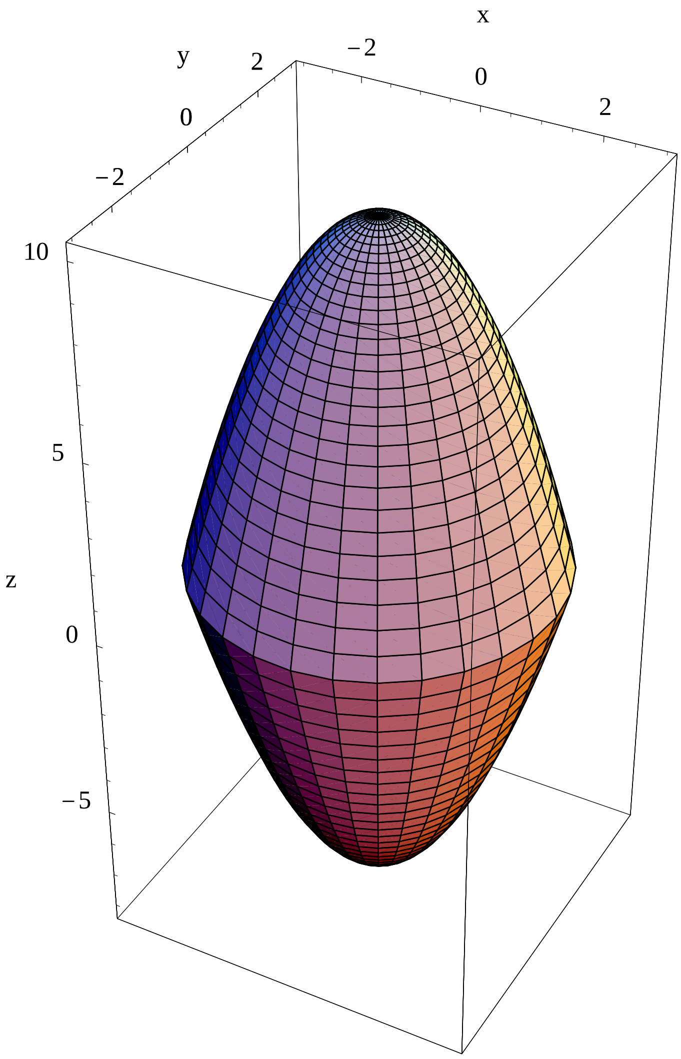

Figure 3.6.12 Region bounded by $z = 10 - x^2 - y^2$ and $z = x^2 + y^2 - 8$

that is, when

$$x^2 + y^2 = 9.$$

Now we may describe the region in the xy-plane described by $x^2 + y^2 \le 9$ as the set of points (x, y) for which $-3 \le x \le 3$ and, for every such fixed x,

$$-\sqrt{3 - x^2} \le y \le \sqrt{3 - x^2}.$$

Moreover, once we have fixed x and y so that (x, y) is inside the circle $x^2 + y^2 = 9$, then (x, y, z) is in D provided $x^2 + y^2 - 8 \le z \le 10 - x^2 - y^2$. Hence we have

$$
\begin{aligned}
V &= \int\int\int_D dxdydz \\
&= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{x^2+y^2-8}^{10-x^2-y^2} dzdydx \\
&= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} z\Big|_{x^2+y^2-8}^{10-x^2-y^2} dydx \\
&= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (18-2x^2-2y^2)dydx \\
&= \int_{-3}^{3}\left(18y-2x^2y-\frac{2y^3}{3}\right)\Bigg|_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx \\
&= \int_{-3}^{3}\left(36\sqrt{9-x^2}-4x^2\sqrt{9-x^2}-\frac{4}{3}(9-x^2)^{\frac{3}{2}}\right) dx \\
&= \int_{-3}^{3}\sqrt{9-x^2}\left(36-4x^2-\frac{4}{3}(9-x^2)\right) dx \\
&= \frac{8}{3}\int_{-3}^{3}(9-x^2)^{\frac{3}{2}} dx.
\end{aligned}
$$

Using the substitution $x = 3\sin(\theta)$, we have $dx = 3\cos(\theta)d\theta$, and so

$$
\begin{aligned}
V &= \frac{8}{3}\int_{-3}^{3}(9-x^2)^{\frac{3}{2}} dx \\
&= \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(9-9\sin^2(x))^{\frac{3}{2}}(3\cos(\theta))d\theta \\
&= 216\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4(\theta)d\theta \\
&= 216\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{1+\cos(2\theta)}{2}\right)^2 d\theta \\
&= 54\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+2\cos(2\theta)+\cos^2(2\theta))d\theta \\
&= 54\left(\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}+\sin(2\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos(4\theta)}{2}d\theta\right) \\
&= 54\pi+27\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}+\frac{27}{4}\sin(4\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= 81\pi.
\end{aligned}
$$

Problems

1. Evaluate each of the following iterated integrals.
   (a) $\int_1^3 \int_0^2 3xy^2 dydx$ (b) $\int_0^{\frac{\pi}{2}} \int_0^{\pi} 4x\sin(x+y)dydx$
   (c) $\int_{-2}^2 \int_{-1}^1 (4-x^2y^2)dxdy$ (d) $\int_0^2 \int_0^1 e^{x+y} dxdy$
2. Evaluate the following definite integrals over the given rectangles.
   (a) $\int\int_D (y^2-2xy)dxdy$, $D=[0,2]\times[0,1]$ (b) $\int\int_D \frac{1}{(x+y)^2}\,dxdy$, $D=[1,2]\times[1,3]$
   (c) $\int\int_D ye^{-x}dxdy$, $D=[0,1]\times[0,2]$ (d) $\int\int_D \frac{1}{2x+y}\,dxdy$, $D=[1,2]\times[0,1]$
3. For each of the following, evaluate the iterated integrals and sketch the region of integration.
   (a) $\int_0^2 \int_0^y (xy^2-x^2)dxdy$ (b) $\int_0^1 \int_{x^4}^{x^2} (x^2+y^2)dydx$
   (c) $\int_0^2 \int_0^{\sqrt{4-x^2}} (4-x^2-y^2)dydx$ (d) $\int_0^1 \int_0^{y^2} xye^{-x-y}dxdy$
4. Find the volume of the region beneath the graph of $f(x,y)=2+x^2+y^2$ and above the rectangle $D=[-1,1]\times[-2,2]$.
5. Find the volume of the region beneath the graph of $f(x,y)=4-x^2+y^2$ and above the region $D=\{(x,y):0\le x\le 2, -x\le y\le x\}$. Sketch the region D.
6. Evaluate $\int\int_D xydxdy$, where D is the region bounded by the x-axis, the y-axis, and the line $y=2-x$.
7. Evaluate $\int\int_D e^{-x^2}dxdy$ where $D=\{(x,y):0\le y\le 1, y\le x\le 1\}$.
8. Find the volume of the region in $\mathbb{R}^3$ described by $x\ge 0$, $y\ge 0$, and $0\le z\le 4-2y-4x$.
9. Find the volume of the region in $\mathbb{R}^3$ lying above the xy-plane and below the surface with equation $z=16-x^2-y^2$.
10. Find the volume of the region in $\mathbb{R}^3$ lying above the xy-plane and below the surface with equation $z=4-2x^2-y^2$.
11. Evaluate each of the following iterated integrals.
    (a) $\int_1^2 \int_0^3 \int_{-2}^2 (4-x^2-z^2)dydxdz$ (b) $\int_{-2}^3 \int_{-1}^2 \int_0^2 3xyzdxdydz$
    (c) $\int_0^4 \int_0^x \int_0^{x+y} (x^2-yz)dzdydx$ (d) $\int_0^1 \int_0^x \int_0^{x+y} \int_0^{x+y+z} wdwdzdydx$

12. Find the volume of the region in $\mathbb{R}^3$ bounded by the paraboloids with equations $z = 3 - x^2 - y^2$ and $z = x^2 + y^2 - 5$.

13. Evaluate $\int\int\int_D xy\,dx\,dy\,dz$, where D is the region bounded by the xy-plane, the yz-plane, the xz-plane, and the plane with equation $z = 4 - x - y$.

14. If $f(x, y, z)$ represents the density of mass at the point (x, y, z) of an object occupying a region D in $\mathbb{R}^3$, then
$$\int\int\int_D f(x, y, z)\,dx\,dy\,dz$$
is the total mass of the object and the point $(\bar{x}, \bar{y}, \bar{z})$, where
$$\bar{x} = \frac{1}{m}\int\int\int_D xf(x, y, z)\,dx\,dy\,dz,$$
$$\bar{y} = \frac{1}{m}\int\int\int_D yf(x, y, z)\,dx\,dy\,dz,$$
and
$$\bar{z} = \frac{1}{m}\int\int\int_D zf(x, y, z)\,dx\,dy\,dz,$$
is called the *center of mass* of the object. Suppose D is the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $z = 4 - x - 2y$.
    (a) Find the total mass and center of mass for an object occupying the region D with mass density given by $f(x, y, z) = 1$.
    (b) Find the total mass and center of mass for an object occupying the region D with mass density given by $f(x, y, z) = z$.

15. If X and Y are points chosen at random from the interval $[0, 1]$, then the probability that (X, Y) lies in a subset D of the unit square $[0, 1] \times [0, 1]$ is $\int\int_D dx\,dy$.
    (a) Find the probability that $X \leq Y$.
    (b) Find the probability that $X + Y \leq 1$.
    (c) Find the probability that $XY \geq \frac{1}{2}$.

16. If X, Y, and Z are points chosen at random from the interval $[0, 1]$, then the probability that (X, Y, Z) lies in a subset D of the unit cube $[0, 1] \times [0, 1] \times [0, 1]$ is $\int\int\int_D dx\,dy\,dz$.
    (a) Find the probability that $X \leq Y \leq Z$.
    (b) Find the probability that $X + Y + Z \leq 1$.