The Calculus of Functions

of

Several Variables

Section 2.3

Motion Along a Curve

Velocity and acceleration

Consider a particle moving in space so that its position at time t is given by $\mathbf{x}(t)$. We think of $\mathbf{x}(t)$ as moving along a curve C parametrized by a function f, where $f : \mathbb{R} \to \mathbb{R}^n$. Hence we have $\mathbf{x}(t) = f(t)$, or, more simply, $\mathbf{x} = f(t)$. For us, n will always be 2 or 3, but there are physical situations in which it is reasonable to have larger values of n, and most of what we do in this section will apply to those cases equally well. This is also a good time to introduce the Leibniz notation for a derivative, thus writing

$$\frac{d\mathbf{x}}{dt} = Df(t). \tag{2.3.1}$$

At a given time t_0, the vector $\mathbf{x}(t_0+h) - \mathbf{x}(t_0)$ represents the magnitude and direction of the change of position of the particle along C from time t_0 to time $t_0 + h$, as shown in Figure 2.3.1. Dividing by h, we obtain a vector,

$$\frac{\mathbf{x}(t_0+h) - \mathbf{x}(t_0)}{h}, \tag{2.3.2}$$

with the same direction, but with length approximating the average speed of the particle over the time interval from t_0 to $t_0 + h$. Assuming differentiability and taking the limit as h approaches 0, we have the following definition.

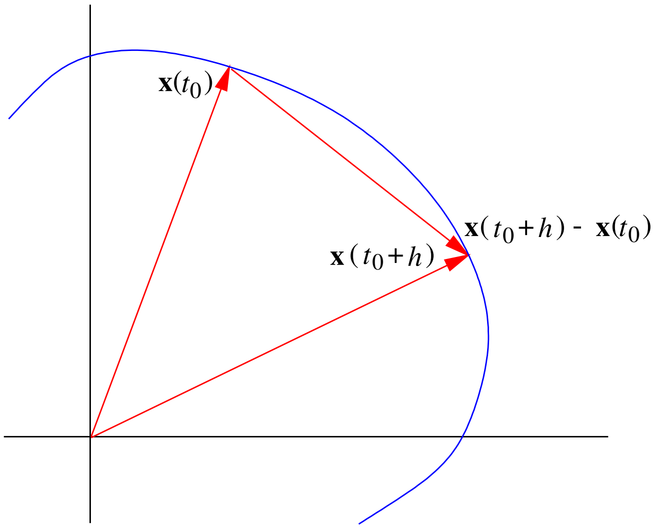

Figure 2.3.1 Motion along a curve C

Definition Suppose $\mathbf{x}(t)$ is the position of a particle at time t moving along a curve C in $\mathbb{R}^n$. We call

$$\mathbf{v}(t) = \frac{d}{dt}\mathbf{x}(t) \tag{2.3.3}$$

the *velocity* of the particle at time t and we call

$$s(t) = \|\mathbf{v}(t)\| \tag{2.3.4}$$

the *speed* of the particle at time t. Moreover, we call

$$\mathbf{a}(t) = \frac{d}{dt}\mathbf{v}(t) \tag{2.3.5}$$

the *acceleration* of the particle at time t.

Example Consider a particle moving along an ellipse so that its position at any time t is

$$\mathbf{x} = (2\cos(t), \sin(t)).$$

Then its velocity is

$$\mathbf{v} = (-2\sin(t), \cos(t)),$$

its speed is

$$s = \sqrt{4\sin^2(t) + \cos^2(t)} = \sqrt{3\sin^2(t) + 1},$$

and its acceleration is

$$\mathbf{a} = (-2\cos(t), -\sin(t)).$$

For example, at $t = \frac{\pi}{4}$ we have

$$\mathbf{x}\Big|_{t=\frac{\pi}{4}} = \left(\sqrt{2}, \frac{1}{\sqrt{2}}\right),$$

$$\mathbf{v}\Big|_{t=\frac{\pi}{4}} = \left(-\sqrt{2}, \frac{1}{\sqrt{2}}\right),$$

$$s\Big|_{t=\frac{\pi}{4}} = \sqrt{\frac{5}{2}},$$

and

$$\mathbf{a}\Big|_{t=\frac{\pi}{4}} = \left(-\sqrt{2}, -\frac{1}{\sqrt{2}}\right).$$

See Figure 2.3.2. Notice that, in this examples, $\mathbf{a} = -\mathbf{x}$ for all values of t.

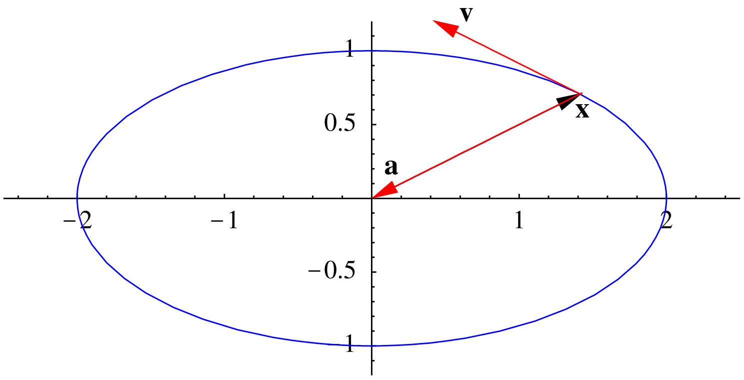

Figure 2.3.2 Position, velocity, and acceleration vectors for motion on an ellipse

Curvature

Suppose $\mathbf{x}$ is the position, $\mathbf{v}$ is the velocity, s is the speed, and $\mathbf{a}$ is the acceleration, at time t, of a particle moving along a curve C. Let $T(t)$ be the unit tangent vector and $N(t)$ be the principal unit normal vector at $\mathbf{x}$. Now

$$T(t) = \frac{\dfrac{d\mathbf{x}}{dt}}{\left\|\dfrac{d\mathbf{x}}{dt}\right\|} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{v}}{s}, \tag{2.3.6}$$

so

$$\mathbf{v} = s\|T(t)\|. \tag{2.3.7}$$

Thus

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\, sT(t) = \frac{ds}{dt}\, T(t) + sDT(t) \tag{2.3.8}$$

Since

$$N(t) = \frac{DT(t)}{\|DT(t)\|}, \tag{2.3.9}$$

we have

$$\mathbf{a} = \frac{ds}{dt}\, T(t) + s\|DT(t)\|N(t). \tag{2.3.10}$$

Note that (2.3.10) expresses the acceleration of a particle as the sum of scalar multiples of the unit tangent vector and the principal unit normal vector. That is,

$$\mathbf{a} = a_T T(t) + a_N N(t), \tag{2.3.11}$$

where

$$a_T = \frac{ds}{dt} \tag{2.3.12}$$

and

$$a_N = s\|DT(t)\|. \tag{2.3.13}$$

However, since $T(t)$ and $N(t)$ are orthogonal unit vectors, we also have

$$\begin{aligned}\mathbf{a}\cdot T(t) &= (a_T T(t) + a_N N(t))\cdot T(t)\\ &= a_T(T(t)\cdot T(t)) + a_N(T(t)\cdot N(t))\\ &= a_T\end{aligned} \tag{2.3.14}$$

and

$$\begin{aligned}\mathbf{a}\cdot N(t) &= (a_T T(t) + a_N N(t))\cdot N(t)\\ &= a_T(T(t)\cdot N(t)) + a_N(N(t)\cdot N(t))\\ &= a_N.\end{aligned} \tag{2.3.15}$$

Hence a_T is the coordinate of $\mathbf{a}$ in the direction of $T(t)$ and a_N is the coordinate of $\mathbf{a}$ in the direction of $N(t)$. Thus (2.3.10) writes the acceleration as a sum of its component in the direction of the unit tangent vector and its component in the direction of the principal unit normal vector. In particular, this shows that the acceleration lies in the plane determined by $T(t)$ and $N(t)$. Moreover, a_T is the rate of change of speed, while a_N is the product of the speed s and $\|DT(t)\|$, the magnitude of the rate of change of the unit tangent vector. Since $\|T(t)\| = 1$ for all t, $\|DT(t)\|$ reflects only the rate at which the direction of $T(t)$ is changing; in other words, $\|DT(t)\|$ is a measurement of how fast the direction of the particle moving along the curve C is changing at time t. If we divide this by the speed of the particle, we obtain a standard measurement of the rate of change of direction of C itself.

Definition Given a curve C with smooth parametrization $\mathbf{x} = f(t)$, we call

$$\kappa = \frac{\|DT(t)\|}{s(t)} \tag{2.3.16}$$

the *curvature* of C at $f(t)$.

Using (2.3.16), we can rewrite (2.3.10) as

$$\mathbf{a} = \frac{ds}{dt}\,T(t) + s^2\kappa N(t). \tag{2.3.17}$$

Hence the coordinate of acceleration in the direction of the tangent vector is the rate of change of the speed and the coordinate of acceleration in the direction of the principal normal vector is the square of the speed times the curvature. Thus the greater the speed or the tighter the curve, the larger the size of the normal component of acceleration; the greater the rate at which speed is increasing, the greater the tangential component of acceleration. This is why drivers are advised to slow down while approaching a curve, and then to accelerate while driving through the curve.

Example Suppose a particle moves along a line in $\mathbb{R}^n$ so that its position at any time t is given by

$$\mathbf{x} = t\mathbf{w} + \mathbf{p},$$

where $\mathbf{w} \neq 0$ and $\mathbf{p}$ are vectors in $\mathbb{R}^n$. Then the particle has velocity

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = \mathbf{w}$$

and speed $s = \|\mathbf{w}\|$, so the unit tangent vector is

$$T(t) = \frac{\mathbf{v}}{s} = \frac{\mathbf{w}}{\|\mathbf{w}\|}.$$

Hence $T(t)$ is a constant vector, so $DT(t) = \mathbf{0}$ and

$$\kappa = \frac{\|DT(t)\|}{s} = 0$$

for all t. In other words, a line has zero curvature, as we should expect since the tangent vector never changes direction.

Example Consider a particle moving along a circle C in $\mathbb{R}^2$ of radius $r > 0$ and center (a, b), with its position at time given by

$$\mathbf{x} = (r\cos(t) + a, r\sin(t) + b).$$

Then its velocity, speed, and acceleration are

$$\mathbf{v} = (-r\sin(t), r\cos(t)),$$

$$s = \sqrt{r^2 \sin^(t) + r^2\cos^2(t)} = r$$

and

$$\mathbf{a} = (-r\cos(t), -r\sin(t)),$$

respectively. Hence the unit tangent vector is

$$T(t) = \frac{\mathbf{v}}{s} = (-\sin(t), \cos(t)).$$

Thus

$$DT(t) = (-\cos(t), -\sin(t))$$

and

$$\|DT(t)\| = \sqrt{\cos^2(t) + \sin^2(t)} = 1.$$

Hence the curvature of C is, for all t,

$$\kappa = \frac{\|DT(t)\|}{s} = \frac{1}{r}.$$

Thus a circle has constant curvature, namely, the reciprocal of the radius of the circle. In particular, the larger the radius of a circle, the smaller the curvature. Also, note that

$$\frac{ds}{dt} = \frac{d}{dt}r = 0,$$

so, from (2.3.10), we have

$$\mathbf{a} = rN(t),$$

which we can verify directly. That is, the acceleration has a normal component, but no tangential component.

Example Now consider a particle moving along an ellipse E so that its position at any time t is

$$\mathbf{x} = (2\cos(t), \sin(t)).$$

Then, as we saw above, the velocity and speed of the particle are

$$\mathbf{v} = (-2\sin(t), \cos(t))$$

and

$$s = \sqrt{3\sin(^2) + 1},$$

respectively. For purposes of differentiation, it will be helpful to rewrite s as

$$s = \sqrt{\frac{3}{2}(1 - \cos(2t)) + 1} = \sqrt{\frac{5 - 3\cos(2t)}{2}}.$$

Then the unit tangent vector is

$$T(t) = \sqrt{\frac{2}{5 - 3\cos(2t)}}\,(-2\sin(t), \cos(t)).$$

Thus

$$DT(t) = \sqrt{\frac{2}{5 - 3\cos(2t)}}\,(-2\cos(t), -\sin(t)) - \frac{3\sqrt{2}\sin(2t)}{(5 - 3\cos(2t))^{\frac{3}{2}}}(-2\sin(t), \cos(t)).$$

So, for example, at $t = \frac{\pi}{4}$, we have

$$\mathbf{x}\big|_{t=\frac{\pi}{4}} = \left(\sqrt{2}, \frac{1}{\sqrt{2}}\right),$$

$$\mathbf{v}\big|_{t=\frac{\pi}{4}} = \left(-\sqrt{2}, \frac{1}{\sqrt{2}}\right),$$

$$s\big|_{t=\frac{\pi}{4}} = \sqrt{\frac{5}{2}},$$

$$T\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{5}}\,(-2,1),$$

$$DT\left(\frac{\pi}{4}\right) = -\frac{1}{5\sqrt{5}}(4,8),$$

and

$$\left\|DT\left(\frac{\pi}{4}\right)\right\| = \frac{1}{5\sqrt{5}}\sqrt{16+64} = \frac{4}{5}.$$

Hence the curvature of E at $\left(\sqrt{2}, \frac{1}{\sqrt{2}}\right)$ is

$$\kappa\Big|_{t=\frac{\pi}{4}} = \frac{\frac{4}{5}}{\sqrt{\frac{5}{2}}} = \frac{4\sqrt{2}}{5\sqrt{5}} = 0.05060,$$

where the final numerical value has been rounded to four decimal places. Although the general expression for κ is complicated, it is easily computed and plotted using a computer algebra system, as shown in Figure 2.3.3. Comparing this with the plot of this ellipse in Figure 2.3.2, we can see why the curvature is greatest around $(2,0)$ and $(-2,0)$, corresponding to $t=0$, $t=\pi$, and $t=2\pi$, and smallest at $(0,1)$ and $(0,-1)$, corresponding to $t=\frac{\pi}{2}$ and $t=\frac{3\pi}{2}$. Finally, as we saw above, the acceleration of the particle is

$$\mathbf{a} = (-2\cos(t), -\sin(t)),$$

so

$$\mathbf{a}\Big|_{t=\frac{\pi}{4}} = \left(-\sqrt{2}, -\frac{1}{\sqrt{2}}\right).$$

Now if we write

$$\mathbf{a}\Big|_{t=\frac{\pi}{4}} = a_T T(t) + a_N N(t),$$

then we may either compute, using (2.3.17),

$$a_T = \frac{ds}{dt}\Big|_{t=\frac{\pi}{4}} = \frac{1}{\sqrt{2}}(5-3\cos(2t))^{-\frac{1}{2}}(3\sin(2t)\Big|_{t=\frac{\pi}{4}} = \frac{3}{\sqrt{10}}$$

and

$$a_N = s^2\Big|_{t=\frac{\pi}{4}} k\Big|_{t=\frac{\pi}{4}} = \frac{5}{2}\frac{4\sqrt{2}}{5\sqrt{5}} = \frac{2\sqrt{2}}{\sqrt{5}} = \frac{4}{\sqrt{10}},$$

or, using (2.3.14) and (2.3.15),

$$a_T = \mathbf{a}\Big|_{t=\frac{\pi}{4}} \cdot T\left(\frac{\pi}{4}\right) = \left(-\sqrt{2}, -\frac{1}{\sqrt{2}}\right) \cdot \frac{1}{\sqrt{5}}\,(-2,1) = \frac{3}{\sqrt{10}}$$

and

$$a_N = \mathbf{a}\Big|_{t=\frac{\pi}{4}} \cdot N\left(\frac{\pi}{4}\right) = \left(-\sqrt{2}, -\frac{1}{\sqrt{2}}\right) \cdot \frac{1}{4\sqrt{5}}(-4,-8) = \frac{4}{\sqrt{10}}.$$

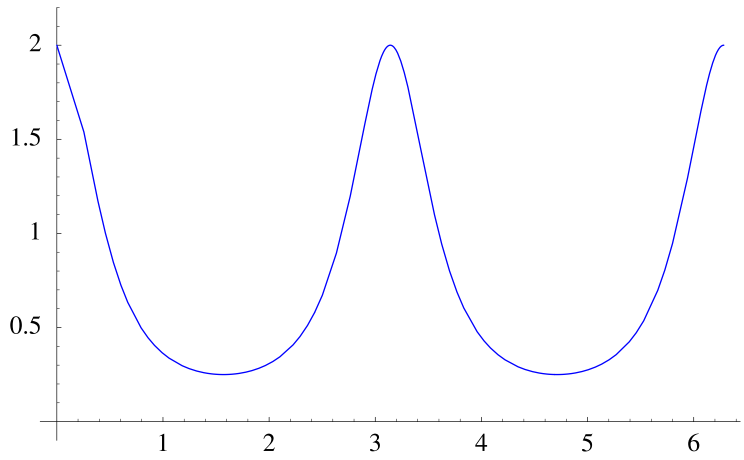

Figure 2.3.3 Curvature of an ellipse

Hence, in either case,

$$\mathbf{a}\Big|_{t=\frac{\pi}{4}} = \frac{3}{\sqrt{10}} T\left(\frac{\pi}{4}\right) + \frac{4}{\sqrt{10}} N\left(\frac{\pi}{4}\right).$$

Arc length

Suppose a particle moves along a curve C in $\mathbb{R}^n$ so that its position at time t is given by $\mathbf{x} = f(t)$ and let D be the distance traveled by the particle from time $t = a$ to $t = b$. We will suppose that $s(t) = \|\mathbf{v}(t)\|$ is continuous on $[a, b]$. To approximate D, we divide $[a, b]$ into n subintervals, each of length

$$\Delta t = \frac{b-a}{n},$$

and label the endpoints of the subintervals $a = t_0, t_1, \ldots, t_n = b$. If Δt is small, then the distance the particle travels during the jth subinterval, $j = 1, 2, \ldots, n$, should be, approximately, $s\Delta t$, an approximation which improves as Δt decreases. Hence, for sufficiently small Δt (equivalently, sufficiently large n),

$$\sum_{j=1}^{n} s(t_{j-1})\Delta t \tag{2.3.18}$$

will provide an approximation as close to D as desired. That is, we should define

$$D = \lim_{n\to\infty} \sum_{j=1}^{n} s(t_{j-1})\Delta t. \tag{2.3.19}$$

But (2.3.18) is a Riemann sum (in particular, a left-hand rule sum) which approximates the definite integral

$$\int_a^b s(t)dt. \tag{2.3.20}$$

Hence the limit in (2.3.19) is the value of the definite integral (2.3.20), and so we have the following definition.

Definition Suppose a particle moves along a curve C in $\mathbb{R}^n$ so that its position at time t is given by $\mathbf{x} = f(t)$. Suppose the velocity $\mathbf{v}(t)$ is continuous on the interval $[a, b]$. Then we define the *distance* traveled by the particle from time $t = a$ to time $t = b$ to be

$$\int_a^b \|\mathbf{v}(t)\| dt. \tag{2.3.21}$$

Note that the distance traveled is the length of the curve C if the particle traverses C exactly once. In that case, we call (2.3.21) the *length* of C. In general, for any t such that the interval $[a, t]$ is in the domain of f, we may calculate

$$\sigma(t) = \int_a^t \|\mathbf{v}(u)\| du, \tag{2.3.22}$$

which we call the *arc length function* for C.

Example Consider the helix H parametrized by

$$f(t) = (\cos(t), \sin(t), t).$$

If we let L denote the length of one complete loop of the helix, then a particle traveling along H according to $\mathbf{x} = f(t)$ will traverse this distance as t goes from 0 to 2π. Since

$$\mathbf{v}(t) = (-\sin(t), \cos(t), 1),$$

we have

$$\|\mathbf{v}(t)\| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}.$$

Hence

$$L = \int_0^{2\pi} \sqrt{2} dt = 2\sqrt{2}\pi.$$

Example Suppose a particle moves along a curve C so that its position at time t is given by

$$\mathbf{x} = ((1 + 2\cos(t))\cos(t), (1 + 2\cos(t))\sin(t)).$$

Then C is the curve in Figure 2.3.4, which is called a *limaçon*. The particle will traverse this curve once as t goes from 0 to 2π. Now

$$\mathbf{v} = (-(1 + 2\cos(t))\sin(t) - 2\sin(t)\cos(t), (1 + 2\cos(t))\cos(t) - 2\sin^2(t)),$$

so

$$\begin{aligned}
\|\mathbf{v}\|^2 &= \mathbf{v}\cdot\mathbf{v} \\
&= (1 + 2\cos(t))^2\sin^2(t) + 4(1 + 2\cos(t))\sin^2(t)\cos(t) + 4\sin^2(t)\cos^2(t) \\
&\qquad + (1 + 2\cos(t))^2\cos^2(t) - 4(1 + 2\cos(t))\sin^2(t)\cos(t) + 4\sin^4(t) \quad , \\
&= (1 + 2\cos(t))^2(\sin^2(t) + \cos^2(t)) + 4\sin^2(t)\cos^2(t) + 4\sin^4(t) \\
&= (1 + 2\cos(t))^2 + 4\sin^2(t)\cos^2(t) + 4\sin^4(t)
\end{aligned}$$

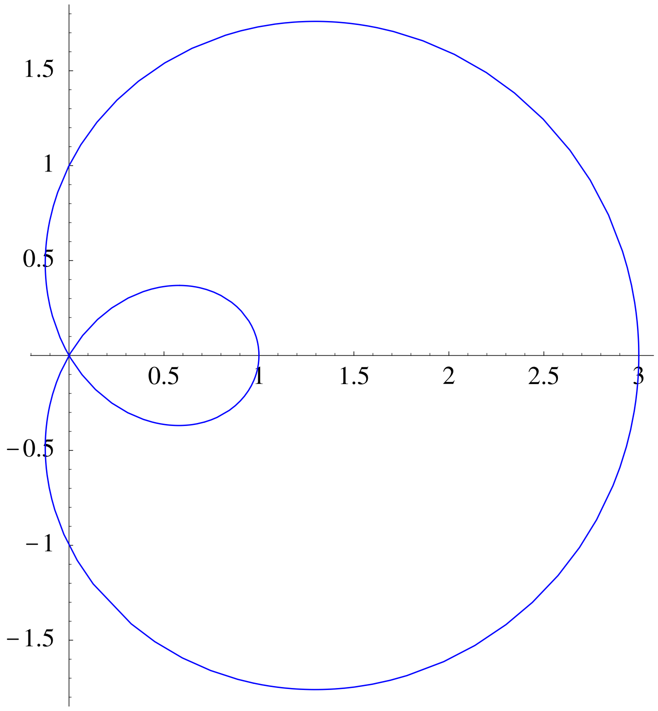

Figure 2.3.4 A limaçon

Hence the length of C is

$$\int_0^{2\pi} \sqrt{(1+2\cos(t))^2 + 4\sin^2(t)\cos^2(t) + 4\sin^4(t)}\, dt = 13.3649,$$

where the integration was performed with a computer and the final result rounded to four decimal places. Note that integrating from 0 to 4π would find the distance the particle travels in going around C twice, namely,

$$\int_0^{4\pi} \sqrt{(1+2\cos(t))^2 + 4\sin^2(t)\cos^2(t) + 4\sin^4(t)}\, dt = 26.7298.$$

Problems

1. For each of the following, suppose a particle is moving along a curve so that its position at time t is given by $\mathbf{x} = f(t)$. Find the velocity and acceleration of the particle.
   (a) $f(t) = (t^2 + 3, \sin(t))$
   (b) $f(t) = (t^2 e^{-2t}, t^3 e^{-2t}, 3t)$
   (c) $f(t) = (\cos(3t^2), \sin(3t^2))$
   (d) $f(t) = (t\cos(t^2), t\sin(t^2), 3t\cos(t^2))$
2. Find the curvature of the following curves at the given point.
   (a) $f(t) = (t, t^2)$, $t = 1$
   (b) $f(t) = (3\cos(t), \sin(t))$, $t = \dfrac{\pi}{4}$

(c) $f(t) = (\cos(t), \sin(t), t)$, $t = \frac{\pi}{3}$

(d) $f(t) = (\cos(t), \sin(t), e^{-t})$, $t = 0$

3. Plot the curvature for each of the following curves over the given interval I.
   (a) $f(t) = (t, t^2)$, $I = [-2, 2]$
   (b) $f(t) = (\cos(t), 3\sin(t))$, $I = [0, 2\pi]$
   (c) $g(t) = ((1 + 2\cos(t))\cos(t), (1 + 2\cos(t))\sin(t))$, $I = [0, 2\pi]$
   (d) $h(t) = (2\cos(t), \sin(t), 2t)$, $I = [0, 2\pi]$
   (e) $f(t) = (4\cos(t) + \sin(4t), 4\sin(t) + \sin(4t))$, $I = [0, 2\pi]$

4. For each of the following, suppose a particle is moving along a curve so that its position at time t is given by $\mathbf{x} = f(t)$. Find the coordinates of acceleration in the direction of the unit tangent vector and in the direction of the principal unit normal vector at the specified point. Write the acceleration as a sum of scalar multiples of the unit tangent vector and the principal unit normal vector.
   (a) $f(t) = (\sin(t), \cos(t))$, $t = \frac{\pi}{3}$
   (b) $f(t) = (\cos(t), 3\sin(t))$, $t = \frac{\pi}{4}$
   (c) $f(t) = (t, t^2)$, $t = 1$
   (d) $f(t) = (\sin(t), \cos(t), t)$, $t = \frac{\pi}{3}$

5. Suppose a particle moves along a curve C in $\mathbb{R}^3$ so that its position at time t is given by $\mathbf{x} = f(t)$. Let $\mathbf{v}$, s, and $\mathbf{a}$ denote the velocity, speed, and acceleration of the particle, respectively, and let κ be the curvature of C.
   (a) Using the facts $\mathbf{v} = sT(t)$ and
   $$\mathbf{a} = \frac{ds}{dt}\,T(t) + s^2\kappa N(t),$$
   show that
   $$\mathbf{v} \times \mathbf{a} = s^3\kappa(T(t) \times N(t)).$$
   (b) Use the result of part (a) to show that
   $$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}.$$

6. Let H be the helix in $\mathbb{R}^3$ parametrized by $f(t) = (\cos(t), \sin(t), t)$. Use the result from Problem 5 to compute the curvature κ of H for any time t.

7. Let C be the elliptical helix in $\mathbb{R}^3$ parametrized by $f(t) = (4\cos(t), 2\sin(t), t)$. Use the result from Problem 5 to compute the curvature κ of C at $t = \frac{\pi}{4}$.

8. Let C be the curve in $\mathbb{R}^2$ which is the graph of the function $\varphi : \mathbb{R} \to \mathbb{R}$. Use the result from Problem 5 to show that the curvature of C at the point $(t, \varphi(t))$ is
   $$\kappa = \frac{|\varphi''(t)|}{(1 + (\varphi'(t))^2)^{\frac{3}{2}}}.$$

9. Let P be the graph of $f(t) = t^2$. Use the result from Problem 8 to find the curvature of P at $(1, 1)$ and $(2, 4)$.

10. Let C be the graph of $f(t) = t^3$. Use the result from Problem 8 to find the curvature of C at $(1, 1)$ and $(2, 8)$.

11. Let C be the graph of $g(t) = \sin(t)$. Use the result from Problem 8 to find the curvature of C at $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

12. For each of the following, suppose a particle is moving along a curve so that its position at time t is given by $\mathbf{x} = f(t)$. Find the distance traveled by the particle over the given time interval.
    (a) $f(t) = (\sin(t), 3\cos(t))$, $I = [0, 2\pi]$
    (b) $f(t) = (\cos(\pi t), \sin(\pi t), 2t)$, $I = [0, 4]$
    (c) $f(t) = (t, t^2)$, $I = [0, 2]$
    (d) $f(t) = (t\cos(t), t\sin(t))$, $I = [0, 2\pi]$
    (e) $f(t) = (\cos(2\pi t), \sin(2\pi t), 3t^2, t)$, $I = [0, 1]$
    (f) $f(t) = (e^{-t}\cos(\pi t), e^{-t}\sin(\pi t))$, $I = [-2, 2]$
    (g) $f(t) = (4\cos(t) + \sin(4t), 4\sin(t) + \sin(4t))$, $I = [0, 2\pi]$

13. Verify that the circumference of a circle of radius r is $2\pi r$.

14. The curve parametrized by

$$f(t) = (\sin(2t)\cos(t), \sin(2t)\sin(t))$$

    has four "petals." Find the length of one of these petals.

15. The curve C parametrized by $h(t) = (\cos^3(t), \sin^3(t))$ is called a hypocycloid (see Figure 2.2.3 in Section 2.3). Find the length of C.

16. Suppose $\varphi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and let C be the part of the graph of φ over the interval $[a, b]$. Show that the length of C is

$$\int_a^b \sqrt{1 + (\varphi'(t))^2}\, dt.$$

17. Use the result from Problem 16 to find the length of one arch of the graph of $f(t) = \sin(t)$.

18. Let $h : \mathbb{R} \to \mathbb{R}^n$ parametrize a curve C. We say C is *parametrized by arc length* if $\|Dh(t)\| = 1$ for all t.
    (a) Let σ be the arc length function for C using the parametrization f and let σ^{-1} be its inverse function. Show that the function $g : \mathbb{R} \to \mathbb{R}^n$ defined by $g(u) = f(\sigma^{-1}(u))$ parametrizes C by arc length.

(b) Let C be the circular helix in $\mathbb{R}^3$ with parametrization $f(t) = (\cos(t), \sin(t), t)$. Find a function $g : \mathbb{R} \to \mathbb{R}^n$ which parametrizes C by arc length.

19. Suppose $f : \mathbb{R} \to \mathbb{R}^n$ is continuous on the closed interval $[a, b]$ and has coordinate functions $f_1, f_2, \ldots, f_n$. We define the *definite integral* of f over the interval $[a, b]$ to be
$$\int_a^b f(t)dt = \left(\int_a^b f_1(t)dt, \int_a^b f_2(t)dt, \ldots, \int_a^b f_n(t)dt \right).$$
Show that if a particle moves so its velocity at time t is $\mathbf{v}(t)$, then, assuming $\mathbf{v}$ is a continuous function on an interval $[a, b]$, the position of the particle for any time t in $[a, b]$ is given by
$$\mathbf{x}(t) = \int_a^t \mathbf{v}(s)ds + \mathbf{x}(a).$$

20. Suppose a particle moves along a curve in $\mathbb{R}^3$ so that its velocity at any time t is
$$\mathbf{v}(t) = (\cos(2t), \sin(2t), 3t).$$
If the particle is at $(0, 1, 0)$ when $t = 0$, use Problem 19 to determine its position for any other time t.

21. Suppose a particle moves along a curve in $\mathbb{R}^3$ so that its acceleration at any time t is
$$\mathbf{a}(t) = (\cos(t), \sin(t), 0).$$
If the particle is at $(1, 2, 0)$ with velocity $(0, 1, 1)$ at time $t = 0$, use Problem 19 to determine its position for any other time t.

22. Suppose a projectile is fired from the ground at an angle α with an initial speed v_0, as shown in Figure 2.3.5. Let $\mathbf{x}(t)$, $\mathbf{v}(t)$, and $\mathbf{a}(t)$ be the position, velocity, and acceleration, respectively, of the projectile at time t.

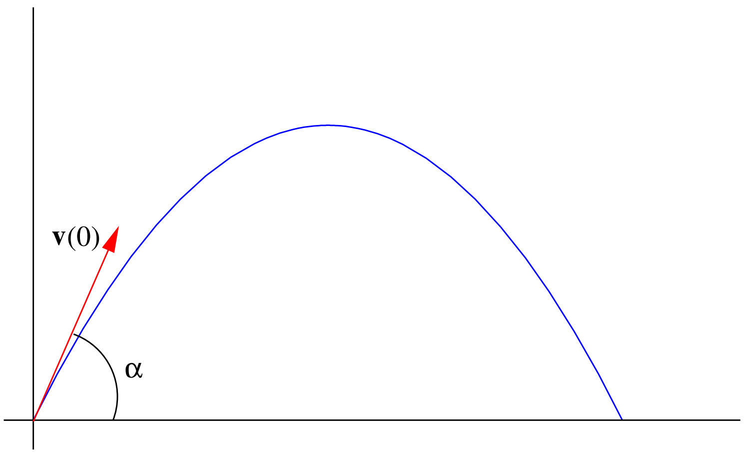

Figure 2.3.5 The path of a projectile

(a) Explain why $\mathbf{x}(0) = (0, 0)$, $\mathbf{v}(0) = (v_0 \cos(\alpha), v_0 \sin(\alpha))$, and $\mathbf{a}(t) = (0, -g)$ for all t, where $g = 9.8$ meters per second per second is the acceleration due to gravity.

(b) Use Problem 19 to find $\mathbf{v}(t)$.

(c) Use Problem 19 to find $\mathbf{x}(t)$.

(d) Show that the curve parametrized by $\mathbf{x}(t)$ is a parabola. That is, let $\mathbf{x}(t) = (x, y)$ and show that $y = ax^2 + bx + c$ for some constants a, b, and c.

(e) Show that the range of the projectile, that is, the horizontal distance traveled, is

$$R = \frac{v_0 \sin(2\alpha)}{g}$$

and conclude that the range is maximized when $\alpha = \frac{\pi}{4}$.

(f) When does the projectile hit the ground?

(g) What is the maximum height reached by the projectile? When does it reach this height?

23. Suppose $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ are unit vectors in $\mathbb{R}^n$, $m \leq n$, which are mutually orthogonal (that is, $a_i \perp a_j$ when $i \neq j$). If $\mathbf{x}$ is a vector in $\mathbb{R}^n$ with

$$\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_m\mathbf{a}_m,$$

show that $x_i = \mathbf{x} \cdot \mathbf{a}_i$, $i = 1, 2, \ldots, m$.