The Calculus of Functions

of

Several Variables

Section 2.1

Curves

Now that we have a basic understanding of the geometry of $\mathbb{R}^n$, we are in a position to start the study of calculus of more than one variable. We will break our study into three pieces. In this chapter we will consider functions $f : \mathbb{R} \to \mathbb{R}^n$, in Chapter 3 we will study functions $f : \mathbb{R}^n \to \mathbb{R}$, and finally in Chapter 4 we will consider the general case of functions $f : \mathbb{R}^m \to \mathbb{R}^n$.

Parametrizations of curves

We begin with some terminology and notation. Given a function $f : \mathbb{R} \to \mathbb{R}^n$, let

$$f_k(t) = k\text{th coordinate of } f(t) \tag{2.1.1}$$

for $k = 1, 2, \ldots, n$. We call $f_k : \mathbb{R} \to \mathbb{R}$ the *kth coordinate function* of f. Note that f_k has the same domain as f and that, for any point t in the domain of f,

$$f(t) = (f_1(t), f_2(t), \ldots, f_n(t)). \tag{2.1.2}$$

If the domain of f is an interval I, then the range of f, that is, the set

$$C = \{\mathbf{x} : \mathbf{x} = f(t) \text{ for some } t \text{ in } I\}, \tag{2.1.3}$$

is called a *curve* with *parametrization* f. The equation $\mathbf{x} = f(t)$, where $\mathbf{x}$ is in $\mathbb{R}^n$, is a *vector equation* for C and, writing $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, the equations

$$\begin{aligned} x_1 &= f_1(t), \\ x_2 &= f_2(t), \\ \vdots\ &\quad \vdots \\ x_n &= f_n(t), \end{aligned} \tag{2.1.4}$$

are *parametric equations* for C.

Example Consider $f : \mathbb{R} \to \mathbb{R}^2$ defined by

$$f(t) = (\cos(t), \sin(t))$$

for $0 \le t \le 2\pi$. Then for every value of t, $f(t)$ is a point on the circle C of radius 1 with center at $(0, 0)$. Note that $f(0) = (1, 0)$, $f\left(\frac{\pi}{2}\right) = (0, 1)$, $f(\pi) = (-1, 0)$, $f\left(\frac{3\pi}{2}\right) = (0, -1)$,

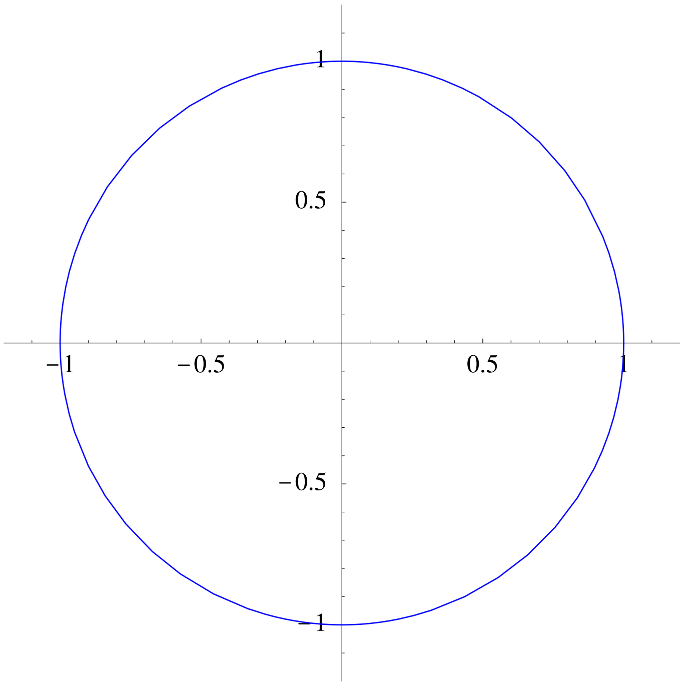

Figure 2.1.1 $f(t) = (\cos(t), \sin(t))$

and $f(2\pi) = (1, 0) = f(0)$. In fact, as t goes from 0 to 2π, $f(t)$ traverses C exactly once in the counterclockwise direction. Thus f is a parametrization of the unit circle C. If we denote a point in $\mathbb{R}^2$ by (x, y), then

$$\begin{aligned} x &= \cos(t), \\ y &= \sin(t), \end{aligned}$$

are parametric equations for C. See Figure 2.1.1. The coordinate functions are

$$\begin{aligned} f_1(t) &= \cos(t), \\ f_2(t) &= \sin(t), \end{aligned}$$

although we frequently write these as simply

$$\begin{aligned} x(t) &= \cos(t), \\ y(t) &= \sin(t). \end{aligned}$$

Example Consider $g : \mathbb{R} \to \mathbb{R}^2$ defined by

$$g(t) = (\sin(2\pi t), \cos(2\pi t))$$

for $0 \leq t \leq 2$. Then g also parametrizes the unit circle C centered at the origin, the same as f in the previous example. However, there is a difference: $g(0) = (0, 1)$, $g\left(\frac{1}{4}\right) = (1, 0)$,

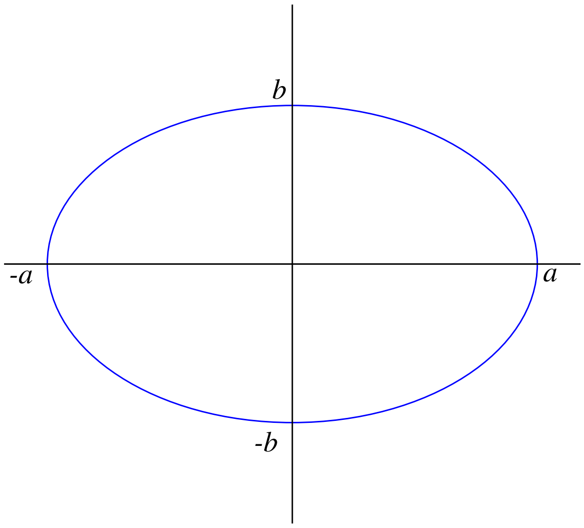

Figure 2.1.2 The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$g\left(\frac{1}{2}\right) = (0, -1)$, $g\left(\frac{3}{4}\right) = (-1, 0)$, and $g(1) = (0, 1) = g(0)$, at which point g starts to repeat its values. Hence $g(t)$, starting at $(0, 1)$, traverses C twice in the clockwise direction as t goes from 0 to 2.

Example More generally, suppose a, b, and α are real numbers, with $a > 0$, $b > 0$, and $\alpha \neq 0$, and let

$$x(t) = a\cos(\alpha t),$$
$$y(t) = b\sin(\alpha t).$$

Then

$$\frac{(x(t))^2}{a^2} + \frac{(y(t))^2}{b^2} = \cos^2(\alpha t) + \sin^2(\alpha t) = 1,$$

so $(x(t), y(t))$ is a point on the ellipse E with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

shown in Figure 2.1.2. Thus the function

$$f(t) = (a\cos(\alpha t), b\cos(\alpha t))$$

parametrizes the ellipse E, traversing the complete ellipse as t goes from 0 to $\left|\frac{2\pi}{\alpha}\right|$.

Example Define $f : \mathbb{R} \to \mathbb{R}^2$ by

$$f(t) = (t\cos(t), t\sin(t))$$

for $-\infty < t < \infty$. Then for negative values of t, $f(t)$ spirals into the origin as t increases, while for positive values of t, $f(t)$ spirals away from the origin. Part of this curve parametrized by f is shown in Figure 2.1.3.

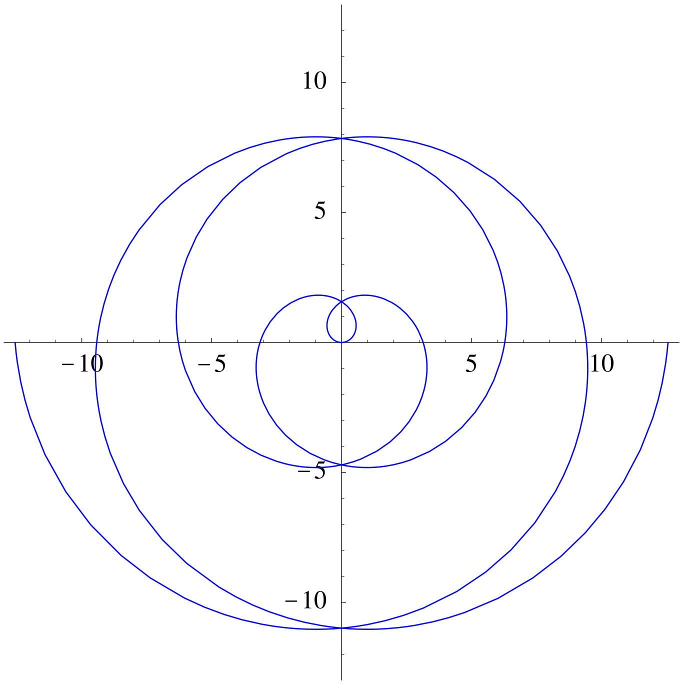

Figure 2.1.3 The spiral $f(t) = (t\cos(t), t\sin(t))$ for $-4\pi \le t \le 4\pi$

Example Define $f : \mathbb{R} \to \mathbb{R}^2$ by

$$f(t) = (3 - 4t, 2 + 3t)$$

for $-\infty < t < \infty$. Then

$$f(t) = t(-4, 3) + (3, 2),$$

so f is a parametrization of the line through the point $(3, 2)$ in the direction of $(-4, 3)$.

In general, a function $f : \mathbb{R} \to \mathbb{R}^n$ defined by $f(t) = t\mathbf{v} + \mathbf{p}$, where $\mathbf{v} \neq 0$ and $\mathbf{p}$ are vectors in $\mathbb{R}^n$, parametrizes a line in $\mathbb{R}^n$.

Example Suppose $g : \mathbb{R} \to \mathbb{R}^3$ is defined by

$$g(t) = (4\cos(t), 4\sin(t), t)$$

for $-\infty < t < \infty$. If we denote the coordinate functions by

$$\begin{aligned} x(t) &= 4\cos(t), \\ y(t) &= 4\sin(t), \\ z(t) &= t, \end{aligned}$$

then

$$(x(t))^2 + (y(t))^2 = 16\cos^2(t) + 16\sin^2(t) = 16.$$

Hence $g(t)$ always lies on a cylinder of radius 1 centered about the z-axis. As t increases, $g(t)$ rises steadily as it winds around this cylinder, completing one trip around the cylinder

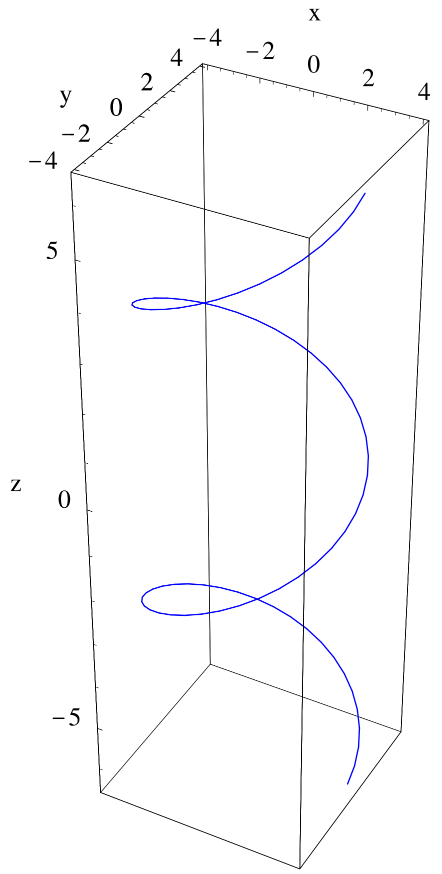

Figure 2.1.4 The helix $f(t) = (4\cos(t), 4\sin(t), t)$, $-2\pi \le t \le 2\pi$

over every interval of length 2π. In other words, g parametrizes a helix, part of which is shown in Figure 2.1.4.

Limits in $\mathbb{R}^n$

As was the case in one-variable calculus, limits are fundamental for understanding ideas such as continuity and differentiability. We begin with the definition of the limit of a sequence of points in $\mathbb{R}^m$.

Definition Let $\{\mathbf{x}_n\}$ be a sequence of points in $\mathbb{R}^m$. We say that the *limit* of $\{\mathbf{x}_n\}$ as n approaches infinity is $\mathbf{a}$, written $\lim\limits_{n\to\infty} \mathbf{x}_n = \mathbf{a}$, if for every $\epsilon > 0$ there is a positive integer N such that

$$\|\mathbf{x}_n - \mathbf{a}\| < \epsilon \tag{2.1.5}$$

whenever $n > N$.

Notice that this definition involves only a slight modification of the definition for the limit of a sequence of real numbers, namely, the use of the norm of a vector instead of the

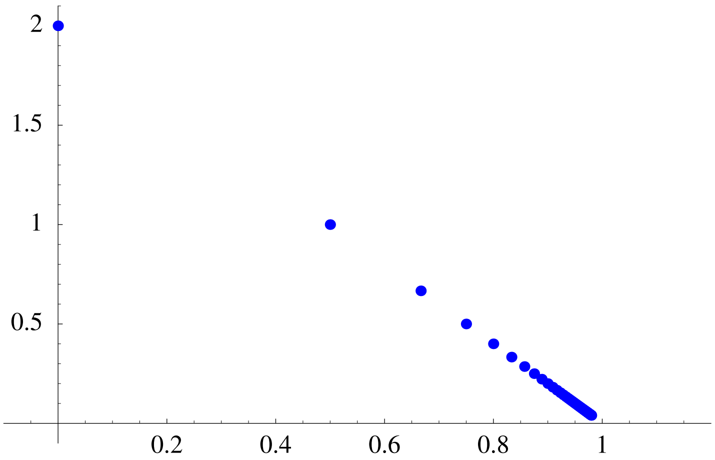

Figure 2.1.5 Points $\left(1-\frac{1}{n},\frac{2}{n}\right)$ approaching $(1,0)$

absolute value of a real number in (2.1.5). In words, $\lim_{n\to\infty}\mathbf{x}_n=\mathbf{a}$ if, given any $\epsilon>0$, we can always find a point in the sequence beyond which all terms of the sequence lie within $B^n(\mathbf{a},\epsilon)$, the open ball of radius ϵ centered at $\mathbf{a}$.

Example Suppose

$$\mathbf{x}_n=\left(1-\frac{1}{n},\frac{2}{n}\right)$$

for $n=1,2,3,\ldots$. Since

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)=1$$

and

$$\lim_{n\to\infty}\frac{2}{n}=0,$$

we should have

$$\lim_{n\to\infty}\mathbf{x}_n=(1,0).$$

To verify this, we first note that

$$\|\mathbf{x}_n-(1,0)\|=\left\|\left(-\frac{1}{n},\frac{2}{n}\right)\right\|=\sqrt{\frac{1}{n^2}+\frac{4}{n^2}}=\frac{\sqrt{5}}{n}.$$

Hence $\|\mathbf{x}_n-(1,0)\|<\epsilon$ whenever $n>\frac{\sqrt{5}}{\epsilon}$. That is, if we let N be any integer greater than or equal to $\frac{\sqrt{5}}{\epsilon}$, then $\|\mathbf{x}_n-(1,0)\|<\epsilon$ whenever $n>N$, verifying that

$$\lim_{n\to\infty}\mathbf{x}_n=(1,0).$$

See Figure 2.1.5.

Put another way, the definition of the limit of a sequence in $\mathbb{R}^m$ says that a sequence $\{\mathbf{x}_n\}$ in $\mathbb{R}^m$ converges to $\mathbf{a}$ in $\mathbb{R}^m$ if and only if the sequence of real numbers $\{\|\mathbf{x}_n-\mathbf{a}\|\}$

converges to 0. That is, $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{a}$ if and only if $\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{a}\| = 0$. Moreover, if we let $\mathbf{x}_n = (x_{n1}, x_{n2}, \ldots, x_{nm})$ and $\mathbf{a} = (a_1, a_2, \ldots, a_m)$, then

$$\|\mathbf{x}_n - \mathbf{a}\| = \sqrt{(x_{n1} - a_1)^2 + (x_{n2} - a_2)^2 + \cdots + (x_{nm} - a_m)^2}, \tag{2.1.6}$$

so $\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{a}\| = 0$ if and only if

$$\lim_{n\to\infty} \sqrt{(x_{n1} - a_1)^2 + (x_{n2} - a_2)^2 + \cdots + (x_{nm} - a_m)^2} = 0. \tag{2.1.7}$$

But (2.1.7) can occur only when $\lim_{n\to\infty} (x_{nk} - a_k)^2 = 0$ for $k = 1, 2, \ldots, m$. Hence $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{a}$ if and only if $\lim_{n\to\infty} x_{nk} = a_k$ for $k = 1, 2, \ldots, m$.

Proposition Suppose $\{\mathbf{x}_n\}$ is a sequence in $\mathbb{R}^m$, $\mathbf{x}_n = (x_{n1}, x_{n2}, \ldots, x_{nm})$, and $\mathbf{a} = (a_1, a_2, \ldots, a_m)$. Then $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{a}$ if and only if $\lim_{n\to\infty} x_{nk} = a_k$ for $k = 1, 2, \ldots, m$.

This proposition tells us that to compute the limit of a sequence in $\mathbb{R}^m$, we need only compute the limit of each coordinate separately, thus reducing the problem of computing limits in $\mathbb{R}^m$ to the problem of finding limits of sequences of real numbers.

Example If

$$\mathbf{x}_n = \left(\frac{2-n}{n^2}, \sin\left(\frac{1}{n}\right), \cos\left(\frac{3}{n}\right)\right),$$

$n = 1, 2, 3, \ldots$, then

$$\lim_{n\to\infty} \mathbf{x}_n = \left(\lim_{n\to\infty} \frac{2-n}{n^2}, \lim_{n\to\infty} \sin\left(\frac{1}{n}\right), \lim_{n\to\infty} \cos\left(\frac{3}{n}\right)\right) = (0, 0, 1).$$

We may now define the limit of a function $f : \mathbb{R} \to \mathbb{R}^m$ at a real number c. Notice that the definition is identical to the definition of a limit for a real-valued function $f : \mathbb{R} \to \mathbb{R}$.

Definition Let c be a real number, let I be an open interval containing c, and let $J = \{t : t \text{ is in } I, t \neq c\}$. Suppose $f : \mathbb{R} \to \mathbb{R}^m$ is defined for all t in J. Then we say that the *limit* of $f(t)$ as t approaches c is $\mathbf{a}$, denoted $\lim_{t\to c} f(t) = \mathbf{a}$, if for every sequence of real numbers $\{t_n\}$ in J,

$$\lim_{n\to\infty} f(t_n) = \mathbf{a} \tag{2.1.8}$$

whenever $\lim_{n\to\infty} t_n = c$.

As in one-variable calculus, we may define the limit of $f(t)$ as t approaches c from the right, denoted

$$\lim_{t\to c^+} f(t),$$

by restricting to sequences $\{t_n\}$ with $t_n > c$ for $n = 1, 2, 3, \ldots$, and the limit of $f(t)$ as t approaches c from the left, denoted

$$\lim_{t\to c^-} f(t),$$

by restricting to sequences $\{t_n\}$ with $t_n < c$ for $n = 1, 2, 3, \ldots$. Moreover, the following useful proposition follows immediately from our definition and the previous proposition.

Proposition Suppose $f : \mathbb{R} \to \mathbb{R}^m$ with

$$f(t) = (f_1(t), f_2(t), \ldots, f_m(t)).$$

The for any real number c,

$$\lim_{t \to c} f(t) = (\lim_{t \to c} f_1(t), \lim_{t \to c} f_2(t), \ldots, \lim_{t \to c} f_m(t)). \tag{2.1.9}$$

Hence the problem of computing limits for functions $f : \mathbb{R} \to \mathbb{R}^m$ reduces to the problem of computing limits of the coordinate functions $f_k : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \ldots, m$, a familiar problem from one-variable calculus. The analogous statements for limits from the right and left also hold.

Example If $f(t) = (t^2 - 1, \sin(t), \cos(t))$ is a function from $\mathbb{R}$ to $\mathbb{R}^3$, then, for example,

$$\lim_{t \to \pi} f(t) = \left(\lim_{t \to \pi} (t^2 - 1), \lim_{t \to \pi} \sin(t), \lim_{t \to \pi} \cos(t) \right) = (\pi^2 - 1, 0, -1).$$

Definitions for continuity also follow the pattern of the related definitions in one-variable calculus.

Definition Suppose $f : \mathbb{R} \to \mathbb{R}^m$. We say f is *continuous at a point* c if

$$\lim_{t \to c} f(t) = f(c). \tag{2.1.10}$$

We say f is *continuous from the right at* c if

$$\lim_{t \to c^+} f(t) = f(c) \tag{2.1.11}$$

and *continuous from the left at* c if

$$\lim_{t \to c^-} f(t) = f(c). \tag{2.1.12}$$

We say f is *continuous* on an open interval (a, b) if f is continuous at every point c in (a, b) and we say f is *continuous* on a closed interval $[a, b]$ if f is continuous on the open interval (a, b), continuous from the right at a, and continuous from the left at b.

If $f(t) = (f_1(t), f_2(t), \ldots, f_m(t))$, then f is continuous at a point c if and only if

$$\lim_{t \to c} f(t) = (\lim_{t \to c} f_1(t), \lim_{t \to c} f_2(t), \ldots, \lim_{t \to c} f_m(t) = f(c) = (f_1(c), f_2(c), \ldots, f_m(c)),$$

which is true if and only if $\lim\limits_{t \to c} f_k(t) = f_k(c)$ for $k = 1, 2, \ldots, m$. In other words, we have the following useful proposition.

Proposition A function $f : \mathbb{R} \to \mathbb{R}^m$ with $f(t) = (f_1(t), f_2(t), \ldots, f_m(t))$ is continuous at a point c if and only if the coordinate functions $f_1, f_2, \ldots, f_m$ are each continuous at c.

Similar statements hold for continuity from the right and from the left.

Example The function $f : \mathbb{R} \to \mathbb{R}^3$ defined by

$$f(t) = (\sin(t^2), t^3 + 4, \cos(t))$$

is continuous on the interval $(-\infty, \infty)$ since each of its coordinate functions is continuous on $(-\infty, \infty)$.

Problems

1. Plot the curves parametrized by the following functions over the specified intervals I.
   (a) $f(t) = (3t + 1, 2t - 1)$, $I = [-5, 5]$
   (b) $g(t) = (t, t^2)$, $I = [-3, 3]$
   (c) $f(t) = (3\cos(t), 3\sin(t))$, $I = [0, 2\pi]$
   (d) $h(t) = (3\cos(t), 3\sin(t))$, $I = [0, \pi]$
   (e) $f(t) = (4\cos(2t), 2\sin(2t)$, $I = [0, \pi]$
   (f) $g(t) = (-4\cos(t), 2\sin(t))$, $I = [0, \pi]$
   (g) $h(t) = (t\sin(3t), t\cos(3t))$, $I = [-\pi, \pi]$
2. Plot the curves parametrized by the following functions over the specified intervals I.
   (a) $f(t) = (t + 1, 2t - 1, 3t)$, $I = [-4, 4]$
   (b) $g(t) = (\cos(t), t, \sin(t))$, $I = [0, 4\pi]$
   (c) $f(t) = (t\cos(2t), t\sin(2t), t)$, $I = [-10, 10]$
   (d) $h(t) = (\cos(2t), \sin(2t), \sqrt{t})$, $I = [0, 9]$
3. Plot the curves parametrized by the following functions over the specified intervals I.
   (a) $f(t) = (\cos(4\pi t), \sin(5\pi t))$, $I = [-0.5, 0.5]$
   (b) $f(t) = (\cos(6\pi t), \sin(7\pi t))$, $I = [-0.5, 0.5]$
   (c) $h(t) = (\cos^3(t), \sin^3(t))$, $I = [0, 2\pi]$
   (d) $g(t) = (\cos(2\pi t), \sin(2\pi t), \sin(4\pi t))$, $I = [0, 1]$
   (e) $f(t) = (\sin(4t)\cos(t), \sin(4t)\sin(t))$, $I = [0, 2\pi]$
   (f) $h(t) = ((1 + 2\cos(t))\cos(t), (1 + 2\cos(t))\sin(t))$, $I = [0, 2\pi]$
4. Suppose $g : \mathbb{R} \to \mathbb{R}$ and we define $f : \mathbb{R} \to \mathbb{R}^2$ by $f(t) = (t, g(t))$. Describe the curve parametrized by f.
5. For each of the following, compute $\lim_{n\to\infty} \mathbf{x}_n$.
   (a) $\mathbf{x}_n = \left(\dfrac{n+1}{2n+3}, 3 - \dfrac{1}{n}\right)$ (b) $\mathbf{x}_n = \left(\sin\left(\dfrac{n-1}{n}\right), \cos\left(\dfrac{n-1}{n}\right), \dfrac{n-1}{n}\right)$

(c) $\mathbf{x}_n = \left(\frac{2n-1}{n^2+1}, \frac{3n+4}{n+1}, 4-\frac{6}{n^2}, \frac{6n+1}{2n^2+5}\right)$

6. Let $f : \mathbb{R} \to \mathbb{R}^3$ be defined by

$$f(t) = \left(\frac{\sin(t)}{t}, \cos(t), 3t^2\right).$$

Evaluate the following.

(a) $\lim_{t\to\pi} f(t)$

(b) $\lim_{t\to 1} f(t)$

(c) $\lim_{t\to 0} f(t)$

7. Discuss the continuity of each of the following functions.

(a) $f(t) = (t^2+1, \cos(2t), \sin(3t))$

(b) $g(t) = (\sqrt{t+1}, \tan(t))$

(c) $f(t) = \left(\frac{1}{t^2-1}, \sqrt{1-t^2}, \frac{1}{t}\right)$

(d) $g(t) = (\cos(4t), 1-\sqrt{3t+1}, \sin(5t), \sec(t))$

8. Let $f : \mathbb{R} \to \mathbb{R}^3$ be defined by $f(t) = (t^2, 3t, 2t+1)$. Find

$$\lim_{h\to 0} \frac{f(t+h)-f(t)}{h}.$$