The Calculus of Functions

of

Several Variables

Section 1.2

Angles and the Dot Product

Suppose $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ are two vectors in $\mathbb{R}^2$, neither of which is the zero vector $\mathbf{0}$. Let α and β be the angles between $\mathbf{x}$ and $\mathbf{y}$ and the positive horizontal axis, respectively, measured in the counterclockwise direction. Supposing $\alpha \geq \beta$, let $\theta = \alpha - \beta$. Then θ is the angle between $\mathbf{x}$ and $\mathbf{y}$ measured in the counterclockwise direction, as shown in Figure 1.2.1. From the subtraction formula for cosine we have

$$\cos(\theta) = \cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta). \tag{1.2.1}$$

Now

$$\cos(\alpha) = \frac{x_1}{\|\mathbf{x}\|},$$

$$\cos(\beta) = \frac{y_1}{\|\mathbf{y}\|},$$

$$\sin(\alpha) = \frac{x_2}{\|\mathbf{x}\|},$$

and

$$\sin(\beta) = \frac{y_2}{\|\mathbf{y}\|}.$$

Thus, we have

$$\cos(\theta) = \frac{x_1 y_1}{\|x\|\|y\|} + \frac{x_2 y_2}{\|x\|\|y\|} = \frac{x_1 y_1 + x_2 y_2}{\|x\|\|y\|}. \tag{1.2.2}$$

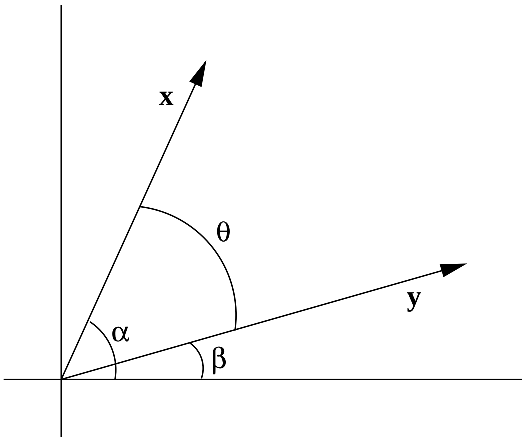

Figure 1.2.1 The angle between two vectors

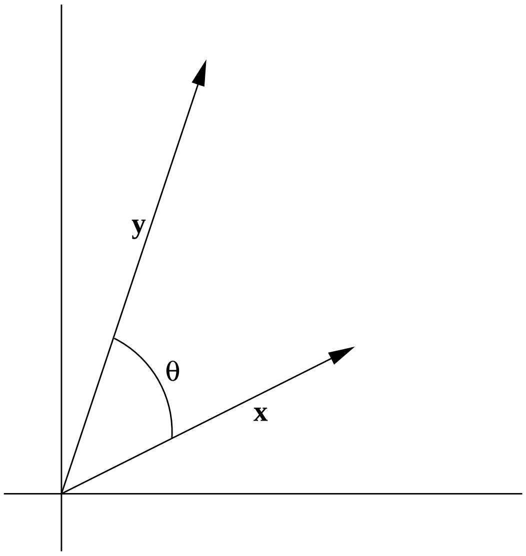

Figure 1.2.2 The angle between $\mathbf{x} = (2, 1)$ and $\mathbf{y} = (1, 3)$

Example Let θ be the smallest angle between $\mathbf{x} = (2, 1)$ and $\mathbf{y} = (1, 3)$, measured in the counterclockwise direction. Then, by (1.2.2), we must have

$$\cos(\theta) = \frac{(2)(1) + (1)(3)}{\|\mathbf{x}\|\|\mathbf{y}\|} = \frac{5}{\sqrt{5}\sqrt{10}} = \frac{1}{\sqrt{2}}.$$

Hence

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}.$$

See Figure 1.2.2.

With more work it is possible to show that if $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ are two vectors in $\mathbb{R}^3$, neither of which is the zero vector $\mathbf{0}$, and θ is the smallest positive angle between $\mathbf{x}$ and $\mathbf{y}$, then

$$\cos(\theta) = \frac{x_1y_1 + x_2y_2 + x_3y_3}{\|\mathbf{x}\|\|\mathbf{y}\|} \tag{1.2.3}$$

The term which appears in the numerators in both (1.2.2) and (1.2.3) arises frequently, so we will give it a name.

Definition If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ are vectors in $\mathbb{R}^n$, then the *dot product* of $\mathbf{x}$ and $\mathbf{y}$, denoted $\mathbf{x} \cdot \mathbf{y}$, is given by

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n. \tag{1.2.4}$$

Note that the dot product of two vectors is a scalar, not another vector. Because of this, the dot product is also called the *scalar product*. It is also an example of what is called an *inner product* and is often denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$.

Example If $\mathbf{x} = (1, 2, -3, -2)$ and $\mathbf{y} = (-1, 2, 3, 5)$, then

$$\mathbf{x} \cdot \mathbf{y} = (1)(-1) + (2)(2) + (-3)(3) + (-2)(5) = -1 + 4 - 9 - 10 = -16.$$

The next proposition lists some useful properties of the dot product.

Proposition For any vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ in $\mathbb{R}^n$ and scalar α,

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}, \tag{1.2.5}$$

$$\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}, \tag{1.2.6}$$

$$(\alpha\mathbf{x}) \cdot \mathbf{y} = \alpha(\mathbf{x} \cdot \mathbf{y}), \tag{1.2.7}$$

$$\mathbf{0} \cdot \mathbf{x} = 0, \tag{1.2.8}$$

$$\mathbf{x} \cdot \mathbf{x} \geq 0, \tag{1.2.9}$$

$$\mathbf{x} \cdot \mathbf{x} = 0 \text{ only if } \mathbf{x} = \mathbf{0}, \tag{1.2.10}$$

and

$$\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2. \tag{1.2.11}$$

These properties are all easily verifiable using the properties of real numbers and the definition of the dot product and will be left to Problem 9 for you to check.

At this point we can say that if $\mathbf{x}$ and $\mathbf{y}$ are two nonzero vectors in either $\mathbb{R}^2$ or $\mathbb{R}^3$ and θ is the smallest positive angle between $\mathbf{x}$ and $\mathbf{y}$, then

$$\cos(\theta) = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}. \tag{1.2.12}$$

We would like to be able to make the same statement about the angle between two vectors in any dimension, but we would first have to define what we mean by the angle between two vectors in $\mathbb{R}^n$ for $n > 3$. The simplest way to do this is to turn things around and use (1.2.12) to define the angle. However, in order for this to work we must first know that

$$-1 \leq \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} \leq 1,$$

since this is the range of values for the cosine function. This fact follows from the following inequality.

Cauchy-Schwarz Inequality For all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$,

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|. \tag{1.2.13}$$

To see why this is so, first note that both sides of (1.2.13) are 0 when $y = \mathbf{0}$, and hence are equal in this case. Assuming $\mathbf{x}$ and $\mathbf{y}$ are fixed vectors in $\mathbb{R}^n$, with $\mathbf{y} \neq \mathbf{0}$, let t be a real number and consider the function

$$f(t) = (\mathbf{x} + t\mathbf{y}) \cdot (\mathbf{x} + t\mathbf{y}). \tag{1.2.14}$$

By (1.2.9), $f(t) \geq 0$ for all t, while from (1.2.6), (1.2.7), and (1.2.11), we see that

$$f(t) = \mathbf{x} \cdot \mathbf{x} + \mathbf{x} \cdot t\mathbf{y} + t\mathbf{y} \cdot \mathbf{x} + t\mathbf{y} \cdot t\mathbf{y} = \|\mathbf{x}\|^2 + 2(\mathbf{x} \cdot \mathbf{y})t + \|\mathbf{y}\|^2 t^2. \tag{1.2.15}$$

Hence f is a quadratic polynomial with at most one root. Since the roots of f are, as given by the quadratic formula,

$$\frac{-2(\mathbf{x} \cdot \mathbf{y}) \pm \sqrt{4(\mathbf{x} \cdot \mathbf{y})^2 - 4\|\mathbf{x}\|^2\|\mathbf{y}\|^2}}{2\|\mathbf{y}\|^2},$$

it follows that we must have

$$4(\mathbf{x} \cdot \mathbf{y})^2 - 4\|\mathbf{x}\|^2\|\mathbf{y}\|^2 \leq 0. \tag{1.2.16}$$

Thus

$$(\mathbf{x} \cdot \mathbf{y})^2 \leq \|\mathbf{x}\|^2\|\mathbf{y}\|^2, \tag{1.2.17}$$

and so

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|. \tag{1.2.18}$$

Note that $|\mathbf{x} \cdot \mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\|$ if and only if there is some value of t for which $f(t) = 0$, which, by (1.2.8) and (1.2.10), happens if and only if $\mathbf{x} + t\mathbf{y} = \mathbf{0}$, that is, $\mathbf{x} = -t\mathbf{y}$, for some value of t. Moreover, if $\mathbf{y} = \mathbf{0}$, then $\mathbf{y} = 0\mathbf{x}$ for any $\mathbf{x}$ in $\mathbb{R}^n$. Hence, in either case, the Cauchy-Schwarz inequality becomes an equality if and only if either $\mathbf{x}$ is a scalar multiple of $\mathbf{y}$ or $\mathbf{y}$ is a scalar multiple of $\mathbf{x}$.

With the Cauchy-Schwarz inequality we have

$$-1 \leq \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} \leq 1 \tag{1.2.19}$$

for any nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$. Thus we may now state the following definition.

Definition If $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in $\mathbb{R}^n$, then we call

$$\theta = \cos^{-1}\left(\frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}\right) \tag{1.2.20}$$

the *angle between* $\mathbf{x}$ and $\mathbf{y}$.

Example Suppose $\mathbf{x} = (1, 2, 3)$ and $\mathbf{y} = (1, -2, 2)$. Then $\mathbf{x} \cdot \mathbf{y} = 1 - 4 + 6 = 3$, $\|\mathbf{x}\| = \sqrt{14}$, and $\|\mathbf{y}\| = 3$, so if θ is the angle between $\mathbf{x}$ and $\mathbf{y}$, we have

$$\cos(\theta) = \frac{3}{3\sqrt{14}} = \frac{1}{\sqrt{14}}.$$

Hence, rounding to four decimal places,

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) = 1.3002.$$

Example Suppose $\mathbf{x} = (2, -1, 3, 1)$ and $\mathbf{y} = (-2, 3, 1, -4)$. Then $\mathbf{x}\cdot\mathbf{y} = -8$, $\|\mathbf{x}\| = \sqrt{15}$, and $\|\mathbf{y}\| = \sqrt{30}$, so if θ is the angle between $\mathbf{x}$ and $\mathbf{y}$, we have, rounding to four decimal places,

$$\theta = \cos^{-1}\left(\frac{-8}{\sqrt{15}\sqrt{30}}\right) = 1.9575.$$

Example Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$ and let α_k, $k = 1, 2, \ldots, n$, be the angle between $\mathbf{x}$ and the kth axis. Then α_k is the angle between $\mathbf{x}$ and the standard basis vector $\mathbf{e}_k$. Thus

$$\cos(\alpha_k) = \frac{\mathbf{x}\cdot\mathbf{e}_k}{\|\mathbf{x}\|\|\mathbf{e}_k\|} = \frac{x_k}{\|\mathbf{x}\|}.$$

That is, $\cos(\alpha_1), \cos(\alpha_2), \ldots, \cos(\alpha_n)$ are the direction cosines of $\mathbf{x}$ as defined in Section 1.1. For example, if $\mathbf{x} = (3, 1, 2)$in $\mathbb{R}^3$, then $\|\mathbf{x}\| = \sqrt{14}$ and the direction cosines of $\mathbf{x}$ are

$$\cos(\alpha_1) = \frac{3}{\sqrt{14}},$$

$$\cos(\alpha_2) = \frac{1}{\sqrt{14}},$$

and

$$\cos(\alpha_3) = \frac{2}{\sqrt{14}},$$

giving us, to four decimal places,

$$\alpha_1 = 0.6405,$$

$$\alpha_2 = 1.3002,$$

and

$$\alpha_3 = 1.0069.$$

Note that if $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in $\mathbb{R}^n$ with $\mathbf{x}\cdot\mathbf{y} = 0$, then the angle between $\mathbf{x}$ and $\mathbf{y}$ is

$$\cos^{-1}(0) = \frac{\pi}{2}.$$

This is the motivation behind our next definition.

Definition Vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ are said to be *orthogonal* (or *perpendicular*), denoted $\mathbf{x} \perp \mathbf{y}$, if $\mathbf{x}\cdot\mathbf{y} = 0$.

It is a convenient convention of mathematics not to restrict the definition of orthogonality to nonzero vectors. Hence it follows from the definition, and (1.2.8), that $\mathbf{0}$ is orthogonal to every vector in $\mathbb{R}^n$. Moreover, $\mathbf{0}$ is the only vector in $\mathbb{R}^n$ which has this property, a fact you will be asked to verify in Problem 12.

Example The vectors $\mathbf{x} = (-1, -2)$ and $\mathbf{y} = (1, 2)$ are both orthogonal to $\mathbf{z} = (2, -1)$ in $\mathbb{R}^2$. Note that $\mathbf{y} = -\mathbf{x}$ and, in fact, any scalar multiple of $\mathbf{x}$ is orthogonal to $\mathbf{z}$.

Example In $\mathbb{R}^4$, $\mathbf{x} = (1, -1, 1, -1)$ is orthogonal to $\mathbf{y} = (1, 1, 1, 1)$. As in the previous example, any scalar multiple of $\mathbf{x}$ is orthogonal to $\mathbf{y}$.

Definition We say vectors $\mathbf{x}$ and $\mathbf{y}$ are *parallel* if $\mathbf{x} = \alpha\mathbf{y}$ for some scalar $\alpha \neq 0$.

This definition says that vectors are parallel when one is a nonzero scalar multiple of the other. From our proof of the Cauchy-Schwarz inequality we know that it follows that if $\mathbf{x}$ and $\mathbf{y}$ are parallel, then $|\mathbf{x}\cdot\mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\|$. Thus if θ is the angle between $\mathbf{x}$ and $\mathbf{y}$,

$$\cos(\theta) = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} = \pm 1.$$

That is, $\theta = 0$ or $\theta = \pi$. Put another way, $\mathbf{x}$ and $\mathbf{y}$ either point in the same direction or they point in opposite directions.

Example The vectors $\mathbf{x} = (1, -3)$ and $\mathbf{y} = (-2, 6)$ are parallel since $\mathbf{x} = -\frac{1}{2}\mathbf{y}$. Note that $\mathbf{x}\cdot\mathbf{y} = -20$ and $\|\mathbf{x}\|\|\mathbf{y}\| = \sqrt{10}\sqrt{40} = 20$, so $\mathbf{x}\cdot\mathbf{y} = -\|\mathbf{x}\|\|\mathbf{y}\|$. It follows that the angle between $\mathbf{x}$ and $\mathbf{y}$ is π.

Two basic results about triangles in $\mathbb{R}^2$ and $\mathbb{R}^3$ are the triangle inequality (the sum of the lengths of two sides of a triangle is greater than or equal to the length of the third side) and the Pythagorean theorem (the sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of the other side). In terms of vectors in $\mathbb{R}^n$, if we picture a vector $\mathbf{x}$ with its tail at the origin and a vector $\mathbf{y}$ with its tail at the tip of $\mathbf{x}$ as two sides of a triangle, then the remaining side is given by the vector $\mathbf{x}+\mathbf{y}$. Thus the triangle inequality may be stated as follows.

Triangle inequality If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$, then

$$\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|. \tag{1.2.21}$$

The first step in verifying (1.2.21) is to note that, using (1.2.11) and (1.2.6),

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) \\ &= \mathbf{x}\cdot\mathbf{x} + 2(\mathbf{x}\cdot\mathbf{y}) + \mathbf{y}\cdot\mathbf{y} \\ &= \|\mathbf{x}\|^2 + 2(\mathbf{x}\cdot\mathbf{y}) + \|\mathbf{y}\|^2. \end{aligned} \tag{1.2.22}$$

Since $\mathbf{x}\cdot\mathbf{y} \leq \|\mathbf{x}\|\|\mathbf{y}\|$ by the Cauchy-Schwarz inequality, it follows that

$$\|\mathbf{x}+\mathbf{y}\|^2 \leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2,$$

from which we obtain the triangle inequality by taking square roots.

Note that in (1.2.22) we have

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

if and only if $\mathbf{x}\cdot\mathbf{y} = 0$, that is, if and only if $\mathbf{x}\perp\mathbf{y}$. Hence we have the following famous result.

Pythagorean theorem Vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ are orthogonal if and only if

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2. \tag{1.2.23}$$

Perhaps the most important application of the dot product is in finding the orthogonal projection of one vector onto another. This is illustrated in Figure 1.2.3, where $\mathbf{w}$ represents the projection of $\mathbf{x}$ onto $\mathbf{y}$. The result of the projection is to break $\mathbf{x}$ into the sum of two *components*, $\mathbf{w}$, which is parallel to $\mathbf{y}$, and $\mathbf{x}-\mathbf{w}$, which is orthogonal to $\mathbf{y}$, a procedure which is frequently very useful. To compute $\mathbf{w}$, note that if θ is the angle between $\mathbf{x}$ and $\mathbf{y}$, then

$$\|\mathbf{w}\| = \|\mathbf{x}\||\cos(\theta)| = \|\mathbf{x}\|\frac{|\mathbf{x}\cdot\mathbf{y}|}{\|\mathbf{x}\|\|\mathbf{y}\|} = \left|\mathbf{x}\cdot\frac{\mathbf{y}}{\|\mathbf{y}\|}\right| = |\mathbf{x}\cdot\mathbf{u}|, \tag{1.2.24}$$

where

$$\mathbf{u} = \frac{\mathbf{y}}{\|\mathbf{y}\|}$$

is the direction of $\mathbf{y}$. Hence $\mathbf{w} = |\mathbf{x}\cdot\mathbf{u}|\mathbf{u}$ when $0 \le \theta \le \frac{\pi}{2}$, which is when $\mathbf{x}\cdot\mathbf{u} > 0$, and $\mathbf{w} = -|\mathbf{x}\cdot\mathbf{u}|\mathbf{u}$ when $\frac{\pi}{2} < \theta \le \pi$, which is when $\mathbf{x}\cdot\mathbf{u} < 0$. Thus, in either case, $\mathbf{w} = (\mathbf{x}\cdot\mathbf{u})\mathbf{u}$.

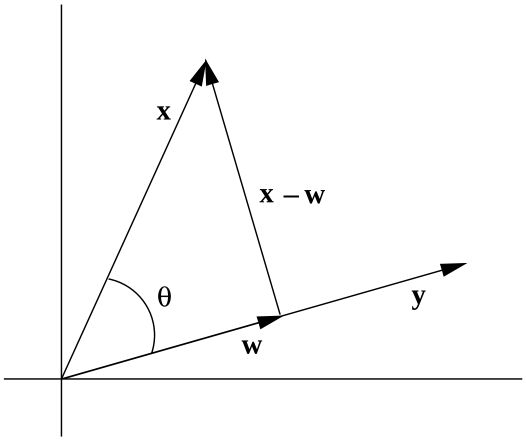

Figure 1.2.3 Orthogonal projection

Definition Given vectors $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{y} \ne \mathbf{0}$, in $\mathbb{R}^n$, the vector

$$\mathbf{w} = (\mathbf{x}\cdot\mathbf{u})\mathbf{u}, \tag{1.2.25}$$

where $\mathbf{u}$ is the direction of $\mathbf{y}$, is called the *orthogonal projection*, or simply *projection*, of $\mathbf{x}$ onto $\mathbf{y}$. We also call $\mathbf{w}$ the *component* of $\mathbf{x}$ in the direction of $\mathbf{y}$ and $\mathbf{x}\cdot\mathbf{u}$ the *coordinate* of $\mathbf{x}$ in the direction of $\mathbf{y}$.

In the special case where $\mathbf{y} = \mathbf{e}_k$, the kth standard basic vector, $k = 1, 2, \ldots, n$, we see that the coordinate of $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in the direction of $\mathbf{y}$ is just $\mathbf{x}\cdot\mathbf{e}_k = x_k$, the kth coordinate of $\mathbf{x}$.

Example Suppose $\mathbf{x} = (1,2,3)$ and $\mathbf{y} = (1,4,0)$. Then the direction of $\mathbf{y}$ is

$$\mathbf{u} = \frac{1}{\sqrt{17}}(1,4,0),$$

so the coordinate of $\mathbf{x}$ in the direction of $\mathbf{y}$ is

$$\mathbf{x} \cdot \mathbf{u} = \frac{1}{\sqrt{17}}(1+8+0) = \frac{9}{\sqrt{17}}.$$

Thus the projection of $\mathbf{x}$ onto $\mathbf{y}$ is

$$\mathbf{w} = \frac{9}{\sqrt{17}}\mathbf{u} = \frac{9}{17}(1,4,0) = \left(\frac{9}{17}, \frac{36}{17}, 0\right).$$

Problems

1. Let $\mathbf{x} = (3,-2)$, $\mathbf{y} = (-2,5)$, and $\mathbf{x} = (4,1)$. Compute each of the following.
   (a) $\mathbf{x} \cdot \mathbf{y}$ (b) $2\mathbf{x} \cdot \mathbf{y}$
   (c) $\mathbf{x} \cdot (3\mathbf{y} - \mathbf{z})$ (d) $-\mathbf{z} \cdot (\mathbf{x} + 5\mathbf{y})$
2. Let $\mathbf{x} = (3,-2,1)$, $\mathbf{y} = (-2,3,5)$, and $\mathbf{z} = (-1,4,1)$. Compute each of the following.
   (a) $\mathbf{x} \cdot \mathbf{y}$ (b) $2\mathbf{x} \cdot \mathbf{y}$
   (c) $\mathbf{x} \cdot (3\mathbf{y} - \mathbf{z})$ (d) $-\mathbf{z} \cdot (\mathbf{x} + 5\mathbf{y})$
3. Let $\mathbf{x} = (3,-2,1,2)$, $\mathbf{y} = (-2,3,4,-5)$, and $\mathbf{z} = (-1,4,1,-2)$. Compute each of the following.
   (a) $\mathbf{x} \cdot \mathbf{y}$ (b) $2\mathbf{x} \cdot \mathbf{y}$
   (c) $\mathbf{x} \cdot (3\mathbf{y} - \mathbf{z})$ (d) $-\mathbf{z} \cdot (\mathbf{x} + 5\mathbf{y})$
4. Find the angles between the following pairs of vectors. First find your answers in radians and then convert to degrees.
   (a) $\mathbf{x} = (1,2)$, $\mathbf{y} = (2,1)$ (b) $\mathbf{z} = (3,1)$, $\mathbf{w} = (-3,1)$
   (c) $\mathbf{x} = (1,1,1)$, $\mathbf{y} = (-1,1,-1)$ (d) $\mathbf{y} = (-1,2,4)$, $\mathbf{z} = (2,3,-1)$
   (e) $\mathbf{x} = (1,2,1,2)$, $\mathbf{y} = (2,1,2,1)$ (f) $\mathbf{x} = (1,2,3,4,5)$, $\mathbf{z} = (5,4,3,2,1)$
5. The three points $(2,1)$, $(1,2)$, and $(-2,1)$ determine a triangle in $\mathbb{R}^2$. Find the measure of its three angles and verify that their sum is π.
6. Given three points $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ in $\mathbb{R}^n$, the vectors $\mathbf{q} - \mathbf{p}$, $\mathbf{r} - \mathbf{p}$, and $\mathbf{q} - \mathbf{r}$ describe the sides of the triangle with vertices at $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$. For each of the following, find the measure of the three angles of the triangle with vertices at the given points.
   (a) $\mathbf{p} = (1,2,1)$, $\mathbf{q} = (-1,-1,2)$, $\mathbf{r} = (-1,3,-1)$
   (b) $\mathbf{p} = (1,2,1,1)$, $\mathbf{q} = (-1,-1,2,3)$, $\mathbf{r} = (-1,3,-1,2)$

7. For each of the following, find the angles between the given vector and the coordinate axes.

   (a) $\mathbf{x} = (-2, 3)$
   (b) $\mathbf{w} = (-1, 2, 1)$
   (c) $\mathbf{y} = (2, 3, 1, -1)$
   (d) $\mathbf{x} = (1, 2, 3, 4, 5)$

8. For each of the following, find the coordinate of $\mathbf{x}$ in the direction of $\mathbf{y}$ and the projection $\mathbf{w}$ of $\mathbf{x}$ onto $\mathbf{y}$. In each case verify that $\mathbf{y} \perp (\mathbf{x} - \mathbf{w})$.

   (a) $\mathbf{x} = (-2, 4)$, $\mathbf{y} = (4, 1)$
   (b) $\mathbf{x} = (4, 1, 4)$, $\mathbf{y} = (-1, 3, 1)$
   (c) $\mathbf{x} = (-4, -3, 1)$, $\mathbf{y} = (1, -1, 6)$
   (d) $\mathbf{x} = (1, 2, 4, -1)$, $\mathbf{y} = (2, -1, 2, 3)$

9. Verify properties (1.2.5) through (1.2.11) of the dot product.

10. If $\mathbf{w}$ is the projection of $\mathbf{x}$ onto $\mathbf{y}$, verify that $\mathbf{y}$ is orthogonal to $\mathbf{x} - \mathbf{w}$.

11. Write $\mathbf{x} = (1, 2, -3)$ as the sum of two vectors, one parallel to $\mathbf{y} = (2, 3, 1)$ and the other orthogonal to $\mathbf{y}$.

12. Suppose $\mathbf{x}$ is a vector with the property that $\mathbf{x} \cdot \mathbf{y} = 0$ for all vectors $\mathbf{y}$ in $\mathbb{R}^n$, $\mathbf{y} \neq \mathbf{x}$. Show that it follows that $\mathbf{x} = \mathbf{0}$.