Sufficient Statistics: Examples

Mathematics 47: Lecture 8

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Example

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- Let X_1, X_2, ..., X_n be a random sample from a Bernoulli distribution with probability of success p.

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- Let X_1, X_2, ..., X_n be a random sample from a Bernoulli distribution with probability of success p.
- Let $T = X_1 + X_2 + \cdots + X_n$ and let f be the joint density of X_1, X_2, ..., X_n.

Example (cont'd)

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- Then

$$
\begin{aligned}
&f(x_1, x_2, \ldots, x_n \mid p) \\
&\quad = \begin{cases} \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}, & \text{if } x_i = 0, 1, i = 1, 2, \ldots, n, \\ 0, & \text{otherwise,} \end{cases} \\
&\quad = \begin{cases} p^{\sum_{i=1}^{n} x_i}(1-p)^{n-\sum_{i=1}^{n} x_i}, & \text{if } x_i = 0, 1, i = 1, 2, \ldots, n, \\ 0, & \text{otherwise,} \end{cases} \\
&\quad = p^t(1-p)^{n-t} h(x_1, x_2, \ldots, x_n) \\
&\quad = g(t, p) h(x_1, x_2, \ldots, x_n),
\end{aligned}
$$

where $g(t, p) = p^t(1-p)^{n-t}$ and

$$
h(x_1, x_2, \ldots, x_n) = \begin{cases} 1, & \text{if } x_i = 0, 1, i =, 2, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}
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&= \begin{cases} \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}, & \text{if } x_i = 0, 1, i = 1, 2, \ldots, n, \\ 0, & \text{otherwise,} \end{cases} \\
&= \begin{cases} p^{\sum_{i=1}^{n} x_i}(1-p)^{n-\sum_{i=1}^{n} x_i}, & \text{if } x_i = 0, 1, i = 1, 2, \ldots, n, \\ 0, & \text{otherwise,} \end{cases} \\
&= p^t(1-p)^{n-t} h(x_1, x_2, \ldots, x_n) \\
&= g(t, p) h(x_1, x_2, \ldots, x_n),
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- Hence T is a sufficient statistic for p.

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- Let $T = X_1 + X_2 + \cdots + X_n$ and let f be the joint density of X_1, X_2, ..., X_n.

Example (cont'd)

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- Then

$$f(x_1, x_2, \ldots, x_n \mid \lambda) = \begin{cases} \prod_{i=1}^{n} \lambda e^{-\lambda x_i}, & \text{if } x_i > 0, i = 1, 2, \ldots, n, \\ 0, & \text{otherwise,} \end{cases}$$
$$= \begin{cases} \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}, & \text{if } x_i > 0, i = 1, 2, \ldots, n, \\ 0, & \text{otherwise,} \end{cases}$$
$$= \lambda^n e^{-\lambda t} h(x_1, x_2, \ldots, x_n)$$
$$= g(t, \lambda) h(x_1, x_2, \ldots, x_n),$$

where

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and

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- Hence T is a sufficient statistic for λ.

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Example (cont'd)

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- Then

$$f(x_1, x_2, \ldots, x_n \mid \theta) = \begin{cases} \prod_{i=1}^{n} \frac{1}{\theta}, & \text{if } 0 < x_i < \theta, i = 1, 2, \ldots, n, \\ 0, & \text{otherwise}, \end{cases}$$

$$= \begin{cases} \frac{1}{\theta^n}, & \text{if } 0 < x_i < \theta, i = 1, 2, \ldots, n, \\ 0, & \text{otherwise}, \end{cases}$$

$$= g(t, \theta) h(x_1, x_2, \ldots, x_n),$$

where

$$g(t, \theta) = \begin{cases} \frac{1}{\theta^n}, & \text{if } \theta > t, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$h(x_1, x_2, \ldots, x_n) = \begin{cases} 1, & \text{if } x_i > 0,\ i = 0, 1, \ldots, n, \\ 0, & \text{otherwise}. \end{cases}$$

Example (con'td)

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- Hence T is a sufficient statistic for θ.

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- Let

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i,$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2,$$

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- Then

$$\begin{aligned} f(x_1, x_2, \ldots, x_n \mid \mu, \sigma^2) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2} \\ &= \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}. \end{aligned}$$

Example (cont'd)

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- Now

$$\begin{aligned}\sum_{i=1}^{n}(x_i-\mu)^2 &= \sum_{i=1}^{n}(x_i-\bar{x}+\bar{x}-\mu)^2 \\ &= \sum_{i=1}^{n}(x_i-\bar{x})^2 + 2\sum_{i=1}^{n}(x_i-\bar{x})(\bar{x}-\mu) + \sum_{i=1}^{n}(\bar{x}-\mu)^2 \\ &= (n-1)s^2 + 2(\bar{x}-\mu)\sum_{i=1}^{n}(x_i-\bar{x}) + n(\bar{x}-\mu)^2 \\ &= (n-1)s^2 + n(\bar{x}-\mu)^2.\end{aligned}$$

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- Thus

$$f(x_1, x_2, \dots, x_n \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} \exp(-\frac{1}{2\sigma^2}((n-1)s^2 + n(\bar{x}-\mu)^2))$$
$$= g(\bar{x}, s^2, \mu, \sigma^2) h(x_1, x_2, \dots, x_n),$$

where

$$g(\bar{x}, s^2, \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} \exp(-\frac{1}{2\sigma^2}((n-1)s^2 + n(\bar{x}-\mu)^2))$$

and

$$h(x_1, x_2, \dots, x_n) = 1.$$

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  and

$$h(x_1, x_2, \ldots, x_n) = 1.$$

- In this case we say $(\bar{X}, S^2)$ is sufficient for (μ, σ^2).

Functions of sufficient statistics

- Note: if T is sufficient for θ and r is a one-to-one function, then $U = r(T)$ is also sufficient for θ since, in that case, for some function g we have
$$L(\theta) = g(t, \theta) = g(r^{-1}(u), \theta).$$

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$$L(\theta) = g(t, \theta) = g(r^{-1}(u), \theta).$$

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If $T = X_1 + X_2 + \cdots + X_n$ is sufficient for θ, then so is
$$\bar{X} = \frac{T}{n}.$$