The Two-Sample t-Test

Mathematics 47: Lecture 30

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- If H_0 is true, and

$$S_p^2 = \frac{(n-1)S_X^2 + (m-1)S_Y^2}{n+m-2},$$

then

$$T = \frac{\bar{X} - \bar{Y}}{S_p\sqrt{\frac{1}{n} + \frac{1}{m}}}$$

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has a t-distribution with $n + m - 2$ degrees of freedom.
- For an observed value t of T, the p-value of the test is $P(T \geq t \mid H_0)$, with appropriate variations for other alternative hypotheses.

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- The proportion of three letter words in the Snodgrass essays were found to be 0.209, 0.205, 0.196, 0.210, 0.202, 0.207, 0.224, 0.223, 0.220, and 0.201, which we assume to be from $N(\mu_Y, \sigma^2)$.

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- We wish to test

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- We compute $\bar{x} = 0.2319$, $s_X^2 = 0.0002121$, $\bar{y} = 0.2097$, $s_Y^2 = 0.00009334$, and

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- Thus we have strong evidence for rejecting H_0.
- The *R* command `> t.test(x,y,var.equal=TRUE)` will perform the above analysis if the data are in the vectors **x** and **y**.

Definition (The F-distribution)

If U and V are independent random variables with distributions $\chi^2(m)$ and $\chi^2(n)$, respectively, then we call the distribution of

$$F = \frac{\dfrac{U}{m}}{\dfrac{V}{n}}$$

an *F-distribution* with m and n degrees of freedom, which we denote $F(m, n)$.

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- Using the R commands `> qf(0.95,4,7)` and `> qf(0.05,7,4)`, respectively, we find $F_{0.95,4,7} = 4.120312$ and $F_{.05,7,4} = 0.2427001$.

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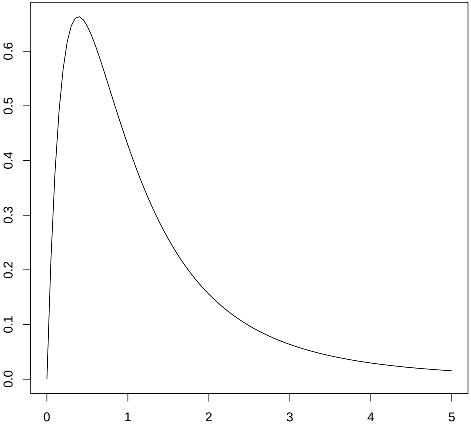
- It may be shown that, if X is $F(m, n)$, then

$$E[X] = \frac{n}{n-2}.$$

Graph of the density of $F(4, 7)$



Comparing variances

- Suppose $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ are independent random samples from $N(\mu_X, \sigma_X^2)$ and $N(\mu_Y, \sigma_Y^2)$, respectively.

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- So

$$\frac{\frac{(n-1)S_X^2}{(n-1)\sigma_X^2}}{\frac{(m-1)S_Y^2}{(m-1)\sigma_Y^2}} = \frac{\sigma_Y^2 S_X^2}{\sigma_X^2 S_Y^2}$$

  is $F(n-1, m-1)$.

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- Hence we should reject H_0 when we observe large values f of F, with p-value $P(F \geq f \mid H_0)$.

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- To test $H_A : \sigma_X^2 < \sigma_Y^2$, we reject H_0 for small observed values f of F, in which case the p-value of the test is $P(F \leq f \mid H_0)$.

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- For the two-sided alternative $H_A : \sigma_X^2 \neq \sigma_Y^2$, we double the appropriate one-sided p-value.

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- Note: If the data are in the vectors **x** and **y**, the *R* command `> var.test(x,y)` will perform the above analysis.