The Neyman-Pearson Lemma

Mathematics 47: Lecture 28

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- *For a fixed* $k > 0$*, let* $\alpha = P(\Lambda \leq k \mid H_0)$ *and*

$$C = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \ : \ \Lambda(x_1, x_2, \ldots, x_n) \leq k\}$$

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- *If* C^* *is any other subset of* $\mathbb{R}^n$ *for which*

$$P((X_1, X_2, \ldots, X_n) \in C^* \mid H_0) \leq \alpha,$$

then

$$P((X_1, X_2, \ldots, X_n) \in C^* \mid H_A) \leq P((X_1, X_2, \ldots, X_n) \in C \mid H_A).$$

Notes on the Neyman-Pearson Lemma

- Note:

$$\begin{aligned}\alpha &= P(\Lambda \leq k \mid H_0) \\ &= P((X_1, X_2, \ldots, X_n) \in C \mid H_0) \\ &= \int \cdots \int_C f_0(x_1, \ldots, x_n) dx_1 \cdots dx_n.\end{aligned}$$

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- The lemma says: among all tests with significance level less than or equal to α, the test with critical region C has the largest power.

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- Then
$$\begin{aligned} P((X_1, X_2, \ldots, X_n) \in C \mid H_A) - P((X_1, X_2, \ldots, X_n) \in C^* \mid H_A) &= \int \cdots \int_C f_1(x_1, \ldots, x_n) dx_1 \cdots dx_n \\ &\quad - \int \cdots \int_{C^*} f_1(x_1, \ldots, x_n) dx_1 \cdots dx_n \\ &= \int \cdots \int_{C \cap (\mathbb{R}^n - C^*)} f_1(x_1, \ldots, x_n) dx_1 \cdots dx_n \\ &\quad - \int \cdots \int_{C^* \cap (\mathbb{R}^n - C)} f_1(x_1, \ldots, x_n) dx_1 \cdots dx_n. \end{aligned}$$

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- And so

$$
\begin{aligned}
P((X_1, X_2, \dots, X_n) \in C \mid H_A) - P((X_1, X_2, \dots, X_n) \in C^* \mid H_A) \\
&\geq \frac{1}{k} \int \cdots \int_{C \cap (\mathbb{R}^n - C^*)} f_0(x_1, \dots, x_n) dx_n \cdots dx_n \\
&\quad - \frac{1}{k} \int \cdots \int_{C^* \cap (\mathbb{R}^n - C)} f_0(x_1, \cdots, x_n) dx_1 \cdots dx_n \\
&= \frac{1}{k} \Big(\int \cdots \int_C f_0(x_1, \dots, x_n) dx_1 \cdots dx_n \\
&\quad - \int \cdots \int_{C^*} f_0(x_1, \dots, x_n) dx_1 \cdots dx_n \Big) \\
&= \frac{1}{k}(\alpha - \alpha^*) \\
&\geq 0.
\end{aligned}
$$

Testing a parameter

- Note: If $X_1, X_2, \ldots, X_n$ is a random sample from a distribution with parameter θ and we wish to test

$$H_0 : \theta = \theta_0$$
$$H_A : \theta = \theta_1,$$

then

$$\Lambda = \frac{L(\theta_0)}{L(\theta_1)},$$

where L is the likelihood function.

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- Let $T = X_1 + X_2 + \cdots + X_n$.

Example (cont'd)

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- Then

$$
\begin{aligned}
\Lambda &= \frac{L(p_0)}{L(p_1)} \\
&= \frac{\prod_{i=1}^n p_0^{X_i}(1-p_0)^{1-X_i}}{\prod_{i=1}^n p_1^{X_i}(1-p_1)^{1-X_i}} \\
&= \frac{p_0^{\sum_{i=1}^n X_i}(1-p_0)^{n-\sum_{i=1}^n X_i}}{p_1^{\sum_{i=1}^n X_i}(1-p_1)^{n-\sum_{i=1}^n X_i}} \\
&= \frac{p_0^T(1-p_0)^{n-T}}{p_1^T(1-p_1)^{n-T}} \\
&= \left(\frac{p_0}{p_1}\right)^T \left(\frac{1-p_0}{1-p_1}\right)^{n-T} \\
&= \left(\frac{1-p_0}{1-p_1}\right)^n \left(\frac{p_0(1-p_1)}{p_1(1-p_0)}\right)^T .
\end{aligned}
$$

Example (cont'd)

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- Now if $p_0 < p_1$, then $1 - p_0 > 1 - p_1$, and so

$$0 < \frac{p_0(1 - p_1)}{p_1(1 - p_0)} < 1.$$

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- Hence when $p_0 < p_1$, the condition $\Lambda \leq k$ is equivalent to the condition $T \geq k^*$ for some k^*.

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- If $p_0 > p_1$, the $1 - p_0 < 1 - p_1$, and so

$$\frac{p_0(1 - p_1)}{p_1(1 - p_0)} > 1.$$

- Hence when $p_0 < p_1$, the condition $\Lambda \leq k$ is equivalent to the condition $T \geq k^*$ for some k^*.
- When $p_0 > p_1$, the condition $\Lambda \leq k$ is equivalent to $T \leq k^*$ for some k^*.

Example (cont'd)

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- That is, if $p_0 < p_1$, the best test is to reject H_0 when $T \geq k^*$, where k^* is chosen so that $P(T \geq k^* \mid p = p_0)$ is the desired level of significance.

Example (cont'd)

- That is, if $p_0 < p_1$, the best test is to reject H_0 when $T \geq k^*$, where k^* is chosen so that $P(T \geq k^* \mid p = p_0)$ is the desired level of significance.
- If $p_0 > p_1$, the best test is to reject H_0 when $T \leq k^*$, where k^* is chosen so that $P(T \leq k^* \mid p = p_0)$ is the desired level of significance.

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- To test

$$H_0 : p = p_0$$
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we find k such that $P(T \geq k \mid p = p_0) = \alpha$ and then reject H_0 if the observed value of T is greater than or equal to k.

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- To test

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we find k such that $P(T \geq k \mid p = p_0) = \alpha$ and then reject H_0 if the observed value of T is greater than or equal to k.

- If π is the power function for this test and π^* is the power function for any other test with significance level less than or equal to α, then

$$\pi(p) \geq \pi^*(p)$$

for all $p \geq p_0$.

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- We say that this test is the *uniformly most powerful (UMP)* test of size α for the hypotheses

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Example (cont'd)

- Hence this test is the best test at significance level α no matter what the true value of p is.
- We say that this test is the *uniformly most powerful (UMP)* test of size α for the hypotheses

$$H_0 : p = p_0$$
$$H_A : p > p_0.$$

- Similarly, the uniformly most powerful test of size α for the hypotheses

$$H_0 : p = p_0$$
$$H_A : p < p_0$$

is to reject H_0 if the observed value of T is less than or equal to some k chosen so that $P(T \leq k \mid p = p_0) = \alpha$.

Example (cont'd)

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- However, there is no uniformly most powerful test for testing

$$H_0 : p = p_0$$
$$H_A : p \neq p_0.$$