Pivotal Quantities

Mathematics 47: Lecture 16

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Pivotal quantities

Definition

Suppose X_1, X_2, ..., X_n is a random sample from a distribution with parameter θ. If $Y = g(X_1, X_2, \ldots, X_n, \theta)$ is a random variable whose distribution does not depend on θ, then we call Y a *pivotal quantity* for θ.

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Example

In finding confidence intervals for μ given a random sample X_1, X_2, ..., X_n from $N(\mu, \sigma^2)$, we used the fact that

$$\frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}}$$

is a pivotal quantity for μ.

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- Then
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is $\chi^2(n-1)$, and so is a pivotal quantity for σ^2.
- Let $\chi^2_{n,\alpha}$ be the α quantile of $\chi^2(n)$.
- Then
$$P\left(\chi^2_{n-1,\frac{\alpha}{2}} \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_{n-1,1-\frac{\alpha}{2}}\right) = 1-\alpha.$$

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- It follows that

$$P\left(\frac{(n-1)S^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2_{n-1,\frac{\alpha}{2}}}\right) = 1-\alpha.$$

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- Hence

$$\left(\frac{(n-1)S^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}, \frac{(n-1)S^2}{\chi^2_{n-1,\frac{\alpha}{2}}}\right)$$

is a $100(1-\alpha)\%$ confidence interal for σ^2.

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- The estimated ages of 19 mineral samples from the Black Forest of Germany were found to be (in millions of years):

249	253	273	260	304
254	269	306	256	283
243	287	303	278	310
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- Then $s^2 = 733.4$.
- We find (either with R, or with Table Va) that $\chi^2_{18,.025} = 8.23$, and $\chi^2_{18,.975} = 31.5$.

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- So, assuming the sample is from $N(\mu, \sigma^2)$, we see that

$$\left(\frac{(18)(733.4)}{31.5}, \frac{(18)(733.4)}{8.23}\right) = (419.1, 1604.0)$$

is a 95% confidence interval for σ^2.

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- Note: Taking square roots, it follows that $(20.47, 40.05)$ is a 95% confidence interval for σ.

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- Recall: the density of $T = X_{(n)}$ is

$$f_T(t) = \begin{cases} \dfrac{n}{\theta^n}\, t^{n-1}, & \text{if } 0 < t < \theta, \\ 0, & \text{otherwise.} \end{cases}$$

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- If we let $Y = \frac{T}{\theta}$, then Y has density

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- Hence Y is a pivotal quantity for θ.

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- Now the distribution function for Y is

$$F_Y(y) = \begin{cases} 0, & \text{if } y \leq 0, \\ y^n, & \text{if } 0 < y \leq 1, \\ 1, & \text{if } y \geq 1. \end{cases}$$

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- So, in particular,

$$P\left(Y \leq (0.95)^{\frac{1}{n}}\right) = F_Y\left((0.95)^{\frac{1}{n}}\right) = 0.95.$$

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- Thus

$$P\left(\frac{X_{(n)}}{(0.95)^{\frac{1}{n}}} \leq \theta\right) = 0.95.$$

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- And so
$$\left(\frac{X_{(n)}}{(0.95)^{\frac{1}{n}}}, \infty\right)$$
is a 95% confidence interval for θ.

One-sided confidence intervals

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Example

- Suppose in a sample of size 20 from a uniform distribution on the interval $(0, \theta)$, we find $x_{(20)} = 945.1132$.
- Then
$$(0.95)^{\frac{1}{20}} \approx 0.9974386,$$
so $(947.5402, \infty)$ is a 95% confidence interval for θ.