Lecture 9: Cardinality

9.1 Binary representations

Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence such that, for each $n = 1, 2, 3, \ldots$, either $a_n = 0$ or $a_n = 1$ and, for any integer N, there exists an integer $n > N$ such that $a_n = 0$. Then

$$0 \leq \frac{a_n}{2^n} \leq \frac{1}{2^n}$$

for $n = 1, 2, 3, \ldots$, so the infinite series

$$\sum_{n=1}^{\infty} \frac{a_n}{2^n}$$

converges to some real number x by the comparison test. Moreover,

$$0 \leq x < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

We call the sequence $\{a_n\}_{n=1}^{\infty}$ the *binary representation* for x, and write

$$x = .a_1 a_2 a_3 a_4 \ldots .$$

Exercise 9.1.1
Suppose $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are both binary representations for x. Show that $a_n = b_n$ for $n = 1, 2, 3, \ldots$.

Now suppose $x \in \mathbb{R}$ with $0 \leq x < 1$. Construct a sequence $\{a_n\}_{n=1}^{\infty}$ as follows: If $0 \leq x < \frac{1}{2}$, let $a_1 = 0$; otherwise, let $a_1 = 1$. For $n = 1, 2, 3, \ldots$, let

$$s_n = \sum_{i=1}^{n} \frac{a_i}{2^n}$$

and set $a_{n+1} = 1$ if

$$s_n + \frac{1}{2^{n+1}} \leq x$$

and $a_{n+1} = 0$ otherwise.

Lemma With the notation as above, $s_n \leq x < s_n + \frac{1}{2^n}$ for $n = 1, 2, 3, \ldots$.

Proof Since

$$s_1 = \begin{cases} 0, & \text{if } 0 \leq x < \frac{1}{2}, \\ \frac{1}{2}, & \text{if } \frac{1}{2} \leq x < 1, \end{cases}$$

it is clear that $s_1 \leq x < s_1 + \frac{1}{2}$. So suppose $n > 1$ and $s_{n-1} \leq x < s_{n-1} + \frac{1}{2^{n-1}}$. If $s_{n-1} + \frac{1}{2^n} \leq x$, then $a_n = 1$ and

$$s_n = s_{n-1} + \frac{1}{2^n} \leq x < s_{n-1} + \frac{1}{2^{n-1}} = s_{n-1} + \frac{1}{2^n} + \frac{1}{2^n} = s_n + \frac{1}{2^n}.$$

If $x < s_{n-1} + \frac{1}{2^n}$, then $a_n = 0$ and

$$s_n = s_{n-1} \leq x < s_{n-1} + \frac{1}{2^n} = s_n + \frac{1}{2^n}.$$

Proposition With the notation as above, $x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$.

Proof Given $\epsilon > 0$, choose an integer N such that $\frac{1}{2^N} < \epsilon$. Then, for any $n > N$, it follows from the lemma that

$$|s_n - x| < \frac{1}{2^n} < \frac{1}{2^N} < \epsilon.$$

Hence

$$x = \lim_{n \to \infty} s_n = \sum_{n=1}^{\infty} \frac{a_n}{2^n}.$$

Lemma With the notation as above, given any integer N there exists an integer $n > N$ such that $a_n = 0$.

Proof If $a_n = 1$ for $n = 1, 2, 3, \ldots$, then

$$x = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1,$$

contradicting the assumption that $0 \leq x < 1$. Now suppose there exists an integer N such that $a_N = 0$ but $a_n = 1$ for every $n > N$. Then

$$x = s_N + \sum_{n=N+1}^{\infty} \frac{1}{2^n} = s_{N-1} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} = s_{N-1} + \frac{1}{2^N},$$

implying that $a_N = 1$, and thus contradicting the assumption that $a_N = 0$.

Combining the previous lemma with the previous proposition yields the following result.

Proposition With the notation as above, $x = .a_1 a_2 a_3 a_4 \ldots$.

Thus we have shown that for every real number $0 \leq x < 1$ there exists a unique binary representation.

9.2 Cardinality

Definition A function $\varphi : A \to B$ is said to be a *one-to-one correspondence* if φ is both one-to-one and onto.

Definition We say sets A and B have the same *cardinality* if there exists a one-to-one correspondence $\varphi : A \to B$.

We denote the fact that A and B have the same cardinality by writing $|A| = |B|$.

Exercise 9.2.1
Define a relation on sets by setting $A \sim B$ if and only if $|A| = |B|$. Show that this relation is an equivalence relation.

Definition Let A be a set. If A has the cardinality of the set $\{1, 2, 3, \ldots, n\}$, $n \in \mathbb{Z}^+$, we say A is *finite* and write $|A| = n$. If A has the cardinality of $\mathbb{Z}^+$, we say A is *countable* and write $|A| = \aleph_0$.

Example If we define $\varphi : \mathbb{Z}^+ \to \mathbb{Z}$ by

$$\varphi(n) = \begin{cases} \frac{n-1}{2}, & \text{if } n \text{ is odd,} \\ -\frac{n}{2}, & \text{if } n \text{ is even,} \end{cases}$$

then φ is a one-to-one correspondence. Thus $|\mathbb{Z}| = \aleph_0$.

Exercise 9.2.2
Let A be the set of even integers. Show that $|A| = \aleph_0$.

Exercise 9.2.3
(a) Let A be a nonempty subset of $\mathbb{Z}^+$. Show that A is either finite or countable.
(b) Let A be a nonempty subset of a countable set B. Show that A is either finite or countable.

Proposition Suppose A and B are countable sets. Then the set $C = A \cup B$ is countable.

Proof Suppose A and B are disjoint, that is, $A \cap B = \emptyset$. Let $\varphi : \mathbb{Z}^+ \to A$ and $\psi : \mathbb{Z}^+ \to B$ be one-to-one correspondences. Define $\tau : \mathbb{Z}^+ \to C$ by

$$\tau(n) = \begin{cases} \varphi\left(\frac{n+1}{2}\right), & \text{if } n \text{ is odd,} \\ \psi\left(\frac{n}{2}\right), & \text{if } n \text{ is even.} \end{cases}$$

Then τ is a one-to-one correspondence, showing that C is countable.

If A and B are not disjoint, then τ is onto but not one-to-one. However, in that case C has the cardinality of an infinite subset of $\mathbb{Z}^+$, and so is countable.

Definition A nonempty set which is not finite is said to be *infinite*. An infinite set which is not countable is said to be *uncountable*.

Exercise 9.2.4
Suppose A is uncountable and $B \subset A$ is countable. Show that $A \setminus B$ is uncountable.

Proposition Suppose A and B are countable. Then $C = A \times B$ is countable.

Proof Let $\varphi : \mathbb{Z}^+ \to A$ and $\psi : \mathbb{Z}^+ \to B$ be one-to-one correspondences. Let $a_i = \varphi(i)$ and $b_i = \psi(i)$. Define $\tau : \mathbb{Z}^+ \to C$ by letting

$$\tau(1) = (a_1, b_1),$$

$$\tau(2) = (a_1, b_2),$$

$$\tau(3) = (a_2, b_1),$$

$$\tau(4) = (a_1, b_3),$$

$$\tau(5) = (a_2, b_2),$$

$$\tau(6) = (a_3, b_1),$$

$$\tau(7) = (a_1, b_4),$$

$$\vdots$$

That is, form the infinite matrix with (a_i, b_j) in the ith row and jth column, and then count the entries by reading down the diagonals from right to left. Then τ is a one-to-one correspondence and C is countable.

Proposition $\mathbb{Q}$ is countable.

Proof By the previous proposition, $\mathbb{Z} \times \mathbb{Z}$ is countable. Let

$$A = \{(p, q) : p, q \in \mathbb{Z}, q > 0, p \text{ and } q \text{ relatively prime}\}.$$

Then A is infinite and $A \subset \mathbb{Z} \times \mathbb{Z}$, so A is countable. But clearly $|\mathbb{Q}| = |A|$, so $\mathbb{Q}$ is countable.

Proposition Suppose for each $i \in \mathbb{Z}^+$, A_i is countable. Then $B = \bigcup_{i=1}^{\infty} A_i$ is countable.

Proof Suppose the sets A_i, $i \in \mathbb{Z}^+$, are pairwise disjoint, that is, $A_i \cap A_j = \emptyset$ for all $i, j \in \mathbb{Z}^+$. For each $i \in \mathbb{Z}^+$, let $\varphi_i : \mathbb{Z}^+ \to A_i$ be a one-to-one correspondence. Then $\psi : \mathbb{Z}^+ \times \mathbb{Z}^+ \to B$ defined by

$$\psi(i, j) = \varphi_i(j)$$

is a one-to-one correspondence, and so $|B| = |\mathbb{Z}^+ \times \mathbb{Z}^+| = \aleph_0$.

If the sets A_i, $i \in \mathbb{Z}^+$, are not disjoint, then ψ is onto but not one-to-one. But then there exists a subset P of $\mathbb{Z}^+ \times \mathbb{Z}^+$ such that $\psi : P \to B$ is a one-to-one correspondence. Since P is an infinite subset of a countable set, P is countable and so $|B| = \aleph_0$.

If in the previous proposition we allow that, for each $i \in \mathbb{Z}^+$, A_i is either finite or countable, then $B = \bigcup_{i=1}^{\infty} A_i$ will be either finite or countable.

Definition Given a set A, the set of all subsets of A is called the *power set* of A, which we denote $\mathcal{P}(A)$.

Example If $A = \{1, 2, 3\}$, then

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}.$$

Proposition If A is finite with $|A| = n$, then $|\mathcal{P}(A)| = 2^n$.

Proof Let

$$B = \{(b_1, b_2, \ldots, b_n) : b_i = 0 \text{ or } b_i = 1, i = 1, 2, \ldots, n\}$$

and let $a_1, a_2, \ldots, a_n$ be the elements of A. Define $\varphi : B \to \mathcal{P}(A)$ by letting

$$\varphi(b_1, b_2, \ldots, b_n) = \{a_i : b_i = 1, i = 1, 2, \ldots, n\}.$$

Then φ is a one-to-one correspondence. The result now follows from the next exercise.

Exercise 9.2.5
With B as in the previous proposition, show that $|B| = 2^n$.

In analogy with the case when A is finite, we let $2^{|A|} = |\mathcal{P}(A)|$ for any nonempty set A.

Definition Suppose A and B are sets for which there exists a one-to-one function $\varphi : A \to B$ but there does not exist a one-to-one correspondence $\psi : A \to B$. Then we write $|A| < |B|$.

Theorem If A is a nonempty set, then $|A| < |\mathcal{P}(A)|$.

Proof Define $\varphi : A \to \mathcal{P}(A)$ by $\varphi(a) = \{a\}$. Then φ is one-to-one. Now suppose $\psi : A \to \mathcal{P}(A)$ is any one-to-one function. Let

$$C = \{a : a \in A, a \notin \psi(a)\}.$$

Suppose there exists $a \in A$ such that $\psi(a) = C$. Then $a \in C$ if and only if $a \notin C$, an obvious contradiction. Thus C is not in the range of ψ, and so ψ is not a one-to-one correspondence.

Lemma Let A be the set of all sequences $\{a_i\}_{i=1}^{\infty}$ with $a_i = 0$ or $a_i = 1$ for each $i = 1, 2, 3, \ldots$. Then $|A| = |\mathcal{P}(\mathbb{Z}^+)|$.

Proof Define $\varphi : A \to \mathcal{P}(\mathbb{Z}^+)$ by

$$\varphi\left((\{a_i\}_{i=1}^{\infty}\right) = \{i : i \in \mathbb{Z}^+, a_i = 1\}.$$

Then φ is a one-to-one correspondence.

Now let B be the set of all sequences $\{a_i\}_{i=1}^{\infty}$ in A such that for every integer N there exists an integer $n > N$ such that $a_n = 0$. Let $C = A \setminus B$,

$$D_0 = \{\{a_i\}_{i=1}^{\infty} : a_i = 1, i = 1, 2, 3, \ldots\},$$

and

$$D_j = \{\{a_i\}_{i=1}^{\infty} : a_j = 0, a_k = 1 \text{ for } k > j\}$$

for $j = 1, 2, 3, \ldots$. Then $|D_0| = 1$ and $|D_j| = 2^{j-1}$ for $j = 1, 2, 3, \ldots$. Moreover,

$$C = \bigcup_{j=0}^{\infty} D_j,$$

so C is countable. Since $A = B \cup C$, and A is uncountable, it follows that B is uncountable. Now if we let

$$I = \{x : x \in \mathbb{R}, 0 \leq x < 1\},$$

we have seen that the function $\varphi : B \to I$ defined by

$$\varphi\left(\{a_i\}_{i=1}^{\infty}\right) = .a_1 a_2 a_3 a_4 \dots$$

is a one-to-one correspondence. It follows that I is uncountable. As a consequence, we have the following result.

Proposition $\mathbb{R}$ is uncountable.

Exercise 9.2.6
Let $I = \{x : x \in \mathbb{R}, 0 \leq x < 1\}$. Show that
(a) $|I| = |\{x : x \in \mathbb{R}, 0 \leq x \leq 1\}|$
(b) $|I| = |\{x : x \in \mathbb{R}, 0 < x < 1\}|$
(c) $|I| = |\{x : x \in \mathbb{R}, 0 < x < 2\}|$
(d) $|I| = |\{x : x \in \mathbb{R}, -1 < x < 1\}|$

Exercise 9.2.7
Let $I = \{x : x \in \mathbb{R}, 0 \leq x < 1\}$ and suppose a and b are real numbers with $a < b$. Show that

$$|I| = |\{x : x \in \mathbb{R}, a \leq x < b\}|.$$

Exercise 9.2.8
Does there exist a set $A \subset \mathbb{R}$ for which $\aleph_0 < |A| < 2^{\aleph_0}$? (Before working too long on this probem, you may wish to read about Cantor's continuum hypothesis.)