# Lecture 8: Infinite Series

## 8.1 Infinite series

**Definition** Given a sequence $\{a_i\}_{i=m}^{\infty}$, let $\{s_n\}_{n=m}^{\infty}$ be the sequence defined by

$$s_n = \sum_{i=m}^{n} a_i.$$

We call the sequence $\{s_n\}_{n=m}^{\infty}$ an *infinite series*. If $\{s_n\}_{n=m}^{\infty}$ converges, we call

$$s = \lim_{n\to\infty} s_n$$

the *sum* of the series. For any integer $n$, $s_n$ is called a *partial sum* of the series.

We will use the notation

$$\sum_{i=m}^{\infty} a_i$$

to denote either $\{s_n\}_{n=m}^{\infty}$, the infinite series, or $s$, the sum of the infinite series. Of course, if $\{s_n\}_{n=m}^{\infty}$ diverges, then we say $\sum_{i=m}^{\infty} a_i$ *diverges*.

**Exercise 8.1.1**
Suppose $\sum_{i=m}^{\infty} a_i$ converges and $\beta \in \mathbb{R}$. Show that $\sum_{i=m}^{\infty} \beta a_i$ also converges and

$$\sum_{i=m}^{\infty} \beta a_i = \beta \sum_{i=m}^{\infty} a_i.$$

**Exercise 8.1.2**
Suppose both $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=m}^{\infty} b_i$ converge. Show that $\sum_{i=m}^{\infty}(a_i + b_i)$ converges and

$$\sum_{i=m}^{\infty}(a_i + b_i) = \sum_{i=m}^{\infty} a_i + \sum_{i=m}^{\infty} b_i.$$

**Exercise 8.1.3**
Given an infinite series $\sum_{i=m}^{\infty} a_i$ and an integer $k \geq m$, show that $\sum_{i=m}^{\infty} a_i$ converges if and only if $\sum_{i=k}^{\infty} a_i$ converges.

**Proposition** Suppose $\sum_{i=m}^{\infty} a_i$ converges. Then $\lim_{n\to\infty} a_n = 0$.

**Proof** Let $s_n = \sum_{i=m}^{n} a_i$ and $s = \lim_{n\to\infty} s_n$. Since $a_n = s_n - s_{n-1}$, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$$

**Exercise 8.1.4**
Let $s = \sum_{i=0}^{\infty}(-1)^n$. Note that

$$s = \sum_{n=0}^{\infty}(-1)^n = 1 - \sum_{n=0}^{\infty}(-1)^n = 1 - s,$$

from which it follows that $s = \frac{1}{2}$. Is this correct?

**Exercise 8.1.5**
Show that for any real number $x \neq 1$,

$$s_n = \sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}$$

(Hint: Note that $x^{n+1} = s_{n+1} - s_n = 1 + xs_n - s_n$.)

**Proposition** For any real number $x$ with $|x| < 1$,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

**Proof** If $s_n = \sum_{i=0}^{n} x^i$, then, by the previous exercise,

$$s_n = \frac{1 - x^{n+1}}{1 - x}.$$

Hence

$$\sum_{n=0}^{\infty} x^n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1-x}.$$

### 8.2 Comparison tests

The following two propositions are together referred to as the *comparison test.*

**Proposition** Suppose $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=k}^{\infty} b_i$ are infinite series for which there exists an integer $N$ such that $0 \leq a_i \leq b_i$ whenever $i \geq N$. If $\sum_{i=k}^{\infty} b_i$ converges, then $\sum_{i=m}^{\infty} a_i$ converges.

**Proof** We need only show that $\sum_{i=N}^{\infty} a_i$ converges. Let $s_n$ be the $n$th partial sum of $\sum_{i=N}^{\infty} a_i$ and let $t_n$ be the $n$th partial sum of $\sum_{i=N}^{\infty} b_i$. Now

$$s_{n+1} - s_n = a_{n+1} \geq 0$$

for every $n \geq N$, so $\{s_n\}_{n=N}^{\infty}$ is a nondecreasing sequence. Moreover,

$$s_n \leq t_n \leq \sum_{i=N}^{\infty} b_i < +\infty$$

for every $n \geq N$. Thus $\{s_n\}_{n=N}^{\infty}$ is a nondecreasing, bounded sequence, and so converges.

**Proposition** Suppose $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=k}^{\infty} b_i$ are infinite series for which there exists an integer $N$ such that $0 \leq a_i \leq b_i$ whenever $i \geq N$. If $\sum_{i=k}^{\infty} a_i$ diverges, then $\sum_{i=m}^{\infty} b_i$ diverges.

**Proof** Again, we need only show that $\sum_{i=N}^{\infty} b_i$ diverges. Let $s_n$ be the $n$th partial sum of $\sum_{i=N}^{\infty} a_i$ and let $t_n$ be the $n$th partial sum of $\sum_{i=N}^{\infty} b_i$. Now $\{s_n\}_{n=N}^{\infty}$ is a nondecreasing sequence which diverges, and so we must have $\lim_{n\to\infty} s_n = +\infty$. Thus given any real number $M$ there exists an integer $L$ such that

$$M < s_n \leq t_n$$

whenever $n > L$. Hence $\lim_{n\to\infty} t_n = +\infty$ and $\sum_{i=m}^{\infty} b_i$ diverges.

**Example** Consider the infinite series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots.$$

Now for $n = 1, 2, 3, \ldots$, we have

$$0 < \frac{1}{n!} \leq \frac{1}{2^{n-1}}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges, it follows that $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges. Moreover,

$$2 < \sum_{n=0}^{\infty} \frac{1}{n!} < 1 + \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{1 - \frac{1}{2}} = 3.$$

We let $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

**Proposition** $e \notin \mathbb{Q}$.

**Proof** Suppose $e = \frac{p}{q}$ where $p, q \in \mathbb{Z}^+$. Let

$$a = q!\left(e - \sum_{i=0}^{q} \frac{1}{n!}\right).$$

Then $a$ is an integer since $q!e = (q-1)!p$ and $n!$ divides $q!$ when $n \leq q$. At the same time

$$\begin{aligned}
0 < a &= q!\left(\sum_{n=0}^{\infty}\frac{1}{n!} - \sum_{i=0}^{q}\frac{1}{n!}\right) \\
&= q!\sum_{n=q+1}^{\infty}\frac{1}{n!} \\
&= \left(\frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \cdots\right) \\
&= \frac{1}{q+1}\left(1 + \frac{1}{q+2} + \frac{1}{(q+2)(q+3)} + \cdots\right) \\
&< \frac{1}{q+1}\left(1 + \frac{1}{q+1} + \frac{1}{(q+1)^2} + \cdots\right) \\
&= \frac{1}{q+1}\sum_{n=0}^{\infty}\frac{1}{(q+1)^n} \\
&= \frac{1}{q+1}\left(\frac{1}{1-\frac{1}{q+1}}\right) = \frac{1}{q}.
\end{aligned}$$

Since this is impossible, we conclude that no such integers $p$ and $q$ exist.

**Definition** We call a real number which is not a rational number an *irrational* number.

**Proposition** Suppose $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=k}^{\infty} b_i$ are infinite series for which there exists an integer $N$ and a real number $M > 0$ such that $0 \leq a_i \leq Mb_i$ whenever $i \geq N$. If $\sum_{i=k}^{\infty} b_i$ converges, then $\sum_{i=m}^{\infty} a_i$ converges.

**Proof** Since $\sum_{i=k}^{\infty} Mb_i$ converges whenever $\sum_{i=k}^{\infty} b_i$ does, the result follows from the comparison test.

**Exercise 8.2.1**
Suppose $\sum_{i=m}^{\infty} a_i$ diverges. Show that $\sum_{i=m}^{\infty} \beta a_i$ diverges for any real number $\beta \neq 0$.

**Proposition** Suppose $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=k}^{\infty} b_i$ are infinite series for which there exists an integer $N$ and a real number $M > 0$ such that $0 \leq a_i \leq Mb_i$ whenever $i \geq N$. If $\sum_{i=m}^{\infty} a_i$ diverges, then $\sum_{i=k}^{\infty} b_i$ diverges.

**Proof** By the comparison test, $\sum_{i=m}^{\infty} Mb_i$ diverges. Hence, by the previous exercise, $\sum_{i=m}^{\infty} b_i$ also diverges.

The results of the next two exercises, which are direct consequences of the last two propositions, are together known as the *limit comparison test*.

**Exercise 8.2.2**
Suppose $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=m}^{\infty} b_i$ are infinite series for which $a_i \geq 0$ and $b_i > 0$ for all $i \geq m$. Show that if $\sum_{i=m}^{\infty} b_i$ converges and

$$\lim_{i\to\infty}\frac{a_i}{b_i} < +\infty,$$

then $\sum_{i=m}^{\infty} a_i$ converges.

**Exercise 8.2.3**
Suppose $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=m}^{\infty} b_i$ are infinite series for which $a_i \geq 0$ and $b_i > 0$ for all $i \geq m$. Show that if $\sum_{i=m}^{\infty} a_i$ diverges and

$$\lim_{i \to \infty} \frac{a_i}{b_i} < +\infty,$$

then $\sum_{i=m}^{\infty} b_i$ diverges.

**Exercise 8.2.4**
Show that

$$\sum_{n=0}^{\infty} \frac{1}{n2^n}$$

converges.

**Exercise 8.2.5**
Show that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges for any real number $x \geq 0$.