Lecture 7: More on Sequences

7.1 Basic theorems about sequences

Proposition Suppose $\{x_i\}_{i \in I}$ is a convergent sequence in $\mathbb{R}$ and $L = \lim_{i \to \infty} x_i$. Then for any real number α, the sequence $\{\alpha x_i\}_{i \in I}$ converges and

$$\lim_{i \to \infty} \alpha x_i = \alpha L.$$

Proof If $\alpha = 0$, then $\{\alpha x_i\}_{i \in I}$ clearly converges to 0. So assume $\alpha \neq 0$. Given $\epsilon > 0$, choose an integer N such that

$$|a_i - L| < \frac{\epsilon}{|\alpha|}$$

whenever $i > N$. Then for any $i > N$ we have

$$|\alpha x_i - \alpha L| = |\alpha||x_i - L| < |\alpha| \frac{\epsilon}{|\alpha|} = \epsilon.$$

Thus $\lim_{i \to \infty} \alpha x_i = \alpha L$.

Proposition Suppose $\{x_i\}_{i \in I}$ and $\{y_i\}_{i \in I}$ are convergent sequences in $\mathbb{R}$ with $L = \lim_{i \to \infty} x_i$ and $M = \lim_{i \to \infty} y_i$. Then the sequence $\{x_i + y_i\}_{i \in I}$ converges and

$$\lim_{i \to \infty} (x_i + y_i) = L + M.$$

Exercise 7.1.1
Prove the previous proposition.

Proposition Suppose $\{x_i\}_{i \in I}$ and $\{y_i\}_{i \in I}$ are convergent sequences in $\mathbb{R}$ with $L = \lim_{i \to \infty} x_i$ and $M = \lim_{i \to \infty} y_i$. Then the sequence $\{x_i y_i\}_{i \in I}$ converges and

$$\lim_{i \to \infty} x_i y_i = LM.$$

Exercise 7.1.2
Prove the previous proposition

Proposition Suppose $\{x_i\}_{i \in I}$ and $\{y_i\}_{i \in I}$ are convergent sequences in $\mathbb{R}$ with $L = \lim_{i \to \infty} x_i$ and $M = \lim_{i \to \infty} y_i$. If $M \neq 0$, then the sequence $\{\frac{x_i}{y_i}\}_{i \in I}$ converges and

$$\lim_{i \to \infty} \frac{x_i}{y_i} = \frac{L}{M}.$$

Proof Since $M \neq 0$ and $M = \lim_{i \to \infty} y_i$, we may choose an integer N such that

$$|y_i| > \frac{|M|}{2}$$

whenever $i > N$. Let B be an upper bound for $\{|x_i| : i \in I\} \cup \{|y_i| : i \in I\}$. Moreover, given any $\epsilon > 0$, we may choose an integer P such that

$$|x_i - L| < \frac{M^2\epsilon}{4B}$$

and

$$|y_i - M| < \frac{M^2\epsilon}{4B}$$

whenever $i > P$. Let K be the larger of N and P. Then, for any $i > K$, we have

$$\begin{aligned}
\left|\frac{x_i}{y_i} - \frac{L}{M}\right| &= \frac{|x_i M - y_i L|}{|y_i M|} \\
&= \frac{|x_i M - x_i y_i + x_i y_i - y_i L|}{|y_i M|} \\
&\leq \frac{|x_i||M - y_i| + |y_i||x_i - L|}{|y_i M|} \\
&< \frac{B\frac{M^2\epsilon}{4B} + B\frac{M^2\epsilon}{4B}}{\frac{M^2}{2}} \\
&= \epsilon.
\end{aligned}$$

Thus

$$\lim_{i\to\infty} \frac{x_i}{y_i} = \frac{L}{M}.$$

Exercise 7.1.3

(a) Show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

(b) Show that

$$\lim_{n\to\infty} \frac{1}{n^2} = 0$$

by (1) using the definition of limit directly and then (2) using previous results.

Exercise 7.1.4

Show that for any positive integer k,

$$\lim_{n\to\infty} \frac{1}{n^k} = 0.$$

Example We may combine the properties of this section to compute

$$\begin{aligned}
\lim_{n\to\infty} \frac{5n^3 + 3n - 6}{2n^3 + 2n^2 - 7} &= \lim_{n\to\infty} \frac{5 + \frac{3}{n^2} - \frac{6}{n^3}}{2 + \frac{2}{n} - \frac{7}{n^3}} \\
&= \frac{\lim_{n\to\infty} 5 + 3\lim_{n\to\infty} \frac{1}{n^2} - 6\lim_{n\to\infty} \frac{1}{n^3}}{\lim_{n\to\infty} 2 + 2\lim_{n\to\infty} \frac{1}{n} - 7\lim_{n\to\infty} \frac{1}{n^3}} \\
&= \frac{5 + 0 + 0}{2 + 0 + 0} = \frac{5}{2}.
\end{aligned}$$

Exercise 7.1.5
Evaluate

$$\lim_{n\to\infty} \frac{3n^5 + 8n^3 - 6n}{8n^5 + 2n^4 - 31},$$

carefully showing each step.

Proposition Suppose $\{x_i\}_{i\in I}$ is a convergent sequence of nonegative real numbers with $L = \lim_{i\to\infty} x_i$. Then the sequence $\{\sqrt{x_i}\}_{i\in I}$ converges and

$$\lim_{i\to\infty} \sqrt{x_i} = \sqrt{L}.$$

Proof Let $\epsilon > 0$ be given. Suppose $L > 0$ and note that

$$|x_i - L| = |\sqrt{x_i} - \sqrt{L}||\sqrt{x_i} + \sqrt{L}|$$

implies that

$$|\sqrt{x_i} - \sqrt{L}| = \frac{|x_i - L|}{|\sqrt{x_i} + \sqrt{L}|}$$

for any $i \in I$. Choose an integer N such that

$$|x_i - L| < \sqrt{L}\epsilon$$

whenever $i > N$. Then, for any $i > N$,

$$|\sqrt{x_i} - \sqrt{L}| = \frac{|x_i - L|}{|\sqrt{x_i} + \sqrt{L}|} < \frac{\sqrt{L}\epsilon}{\sqrt{L}} = \epsilon.$$

Hence $\lim_{i\to\infty} \sqrt{x_i} = \sqrt{L}$.

If $L = 0$, $\lim_{i\to\infty} x_i = 0$, so we may choose an integer N such that $|x_i| < \epsilon^2$ for all $i > N$. Then

$$|\sqrt{x_i}| < \epsilon$$

whenever $i > N$, so $\lim_{i\to\infty} \sqrt{x_i} = 0$.

Exercise 7.1.6
Evaluate

$$\lim_{n\to\infty} \frac{\sqrt{3n^2 + 1}}{5n + 6},$$

carefully showing each step.

Exercise 7.1.7
Given real numbers $r > 0$ and α, show that (a) $\alpha r < r$ if $0 < \alpha < 1$ and (b) $r < \alpha r$ if $\alpha > 1$.

Proposition If $x \in \mathbb{R}$ and $|x| \leq 1$, then

$$\lim_{n\to\infty} x^n = 0.$$

Proof We will first assume $x \geq 0$. Then the sequence $\{x^n\}_{n=1}^{\infty}$ is nonincreasing and bounded below by 0. Hence the sequence converges. Let $L = \lim_{n\to\infty} x^n$. Then

$$L = \lim_{n\to\infty} x^n = x \lim_{n\to\infty} x^{n-1} = xL,$$

from which it follows that $L(1-x) = 0$. Since $1 - x > 0$, we must have $L = 0$. The result for $x < 0$ follows from the next exercise.

Exercise 7.1.8
Show that $\lim_{n\to\infty} |a_n| = 0$ if and only if $\lim_{n\to\infty} a_n = 0$.

7.2 Subsequences

Definition Given a sequence $\{x_i\}_{i=m}^{\infty}$, suppose $\{n_k\}_{k=1}^{\infty}$ is an increasing sequence of integers with

$$m \leq n_1 < n_2 < n_3 < \cdots.$$

Then the sequence $\{x_{n_k}\}_{k=1}^{\infty}$ is called a *subsequence* of $\{x_i\}_{i=m}^{\infty}$.

Example The sequence $\{x_{2k}\}_{k=1}^{\infty}$ is a subsequence of the sequence $\{x_i\}_{i=1}^{\infty}$. For example, $\{\frac{1}{2i}\}_{i=1}^{\infty}$ is a subsequence of $\{\frac{1}{i}\}_{i=1}^{\infty}$.

Exercise 7.2.1
Suppose $\{x_i\}_{i=m}^{\infty}$ converges with $\lim_{i\to\infty} x_i = L$. Show that every subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_i\}_{i=m}^{\infty}$ also converges and $\lim_{k\to\infty} x_{n_k} = L$.

Exercise 7.2.2
Suppose $\{x_i\}_{i=m}^{\infty}$ diverges to $+\infty$. Show that every subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_i\}_{i=m}^{\infty}$ also diverges to $+\infty$.

Exercise 7.2.3
Suppose $\{x_i\}_{i=m}^{\infty}$ diverges to $-\infty$. Show that every subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_i\}_{i=m}^{\infty}$ also diverges to $-\infty$.

Definition Given a sequence $\{x_i\}_{i=m}^{\infty}$, any extended real number λ which is the limit of a subsequence of $\{x_i\}_{i=m}^{\infty}$ is called a *subsequential limit* of $\{x_i\}_{i=m}^{\infty}$.

Example -1 and 1 are both subsequential limits of $\{(-1)^i\}_{i=0}^{\infty}$.

Exercise 7.2.4
Suppose the sequence $\{x_i\}_{i=m}^{\infty}$ is not bounded. Show that either $-\infty$ or $+\infty$ is a subsequential limit of $\{x_i\}_{i=m}^{\infty}$.

Proposition Suppose Λ is the set of all subsequential limits of the sequence $\{x_i\}_{i=m}^{\infty}$. Then $\Lambda \neq \emptyset$.

Proof By the previous exercise, the proposition is true if $\{x_i\}_{i=m}^{\infty}$ is not bounded. So suppose $\{x_i\}_{i=m}^{\infty}$ is bounded and choose real numbers a and b such that $a \leq x_i$ and $b \geq x_i$ for all $i \geq m$. Construct sequences $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ as follows: Let $a_1 = a$ and $b_1 = b$. For $i \geq 1$, let

$$c = \frac{a_{i-1} + b_{i-1}}{2}.$$

If there exists an integer N such that $a_{i-1} \leq x_j \leq c$ for all $j > N$, let $a_i = a_{i-1}$ and $b_i = c$; otherwise, let $a_i = c$ and $b_i = b_{i-1}$. Let $n_1 = m$ and, for $k = 2, 3, 4, \ldots$, let n_k be the smallest integer for which $n_k > n_{k-1}$ and $a_k \leq x_{n_k} \leq b_k$. Then $\{x_{n_k}\}_{k=1}^{\infty}$ is a Cauchy sequence which is a subsequence of $\{x_i\}_{i=m}^{\infty}$. Thus $\{x_{n_k}\}_{k=1}^{\infty}$ converges and $\Lambda \neq \emptyset$.

Exercise 7.2.5
Suppose $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ with $a \leq b$ for every $a \in A$ and $b \in B$. Show that $\sup A \leq \inf B$.

Proposition Let Λ be the set of subsequential limits of a sequence $\{x_i\}_{i=m}^{\infty}$. Then

$$\limsup_{i \to \infty} x_i = \sup \Lambda.$$

Proof Let $s = \sup \Lambda$ and, for $i \geq m$, $u_i = \sup\{x_j : j \geq i\}$. Now since $x_j \leq u_i$ for all $j \geq i$, it follows that $\lambda \leq u_i$ for every $\lambda \in \Lambda$ and $i \geq m$. Hence, from the previous exercise, $s \leq \inf\{u_i : i \geq m\} = \limsup_{i \to \infty} x_i$.

Now suppose $s < \limsup_{i \to \infty} x_i$. Then there exists a real number t such that $s < t < \limsup_{i \to \infty} x_i$. In particular, $t < u_i$ for every $i \geq m$. Let n_1 be the smallest integer for which $n_k \geq m$ and $x_{n_k} > t$. For $k = 2, 3, 4, \ldots$, let n_k be the smallest integer for which $n_k > n_{k-1}$ and $x_{n_k} > t$. Then $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_i\}_{i=m}^{\infty}$ which has a subsequential limit $\lambda \geq t$. Since λ is also then a subsequential limit of $\{x_i\}_{i=m}^{\infty}$, we have $\lambda \in \Lambda$ and $\lambda \geq t > s$, contradicting $s = \sup \Lambda$. Hence we must have $\limsup_{i \to \infty} x_i = \sup \Lambda$.

Exercise 7.2.6
Let Λ be the set of subsequential limits of a sequence $\{x_i\}_{i=m}^{\infty}$. Show that

$$\liminf_{i \to \infty} x_i = \inf \Lambda.$$