Lecture 6: Sequences of Real Numbers

6.1 Limits of sequences

Definition Let $\{a_i\}_{i\in I}$ be a sequence of real numbers. We say $\{a_i\}_{i\in I}$ *converges*, and has *limit* L, if for every real number $\epsilon > 0$ there exists an integer N such that

$$|a_i - L| < \epsilon$$

whenever $i > N$. A sequence $\{a_i\}_{i\in I}$ which does not converge is said to *diverge*.

Definition We say a sequence $\{a_i\}_{i\in I}$ is *nondecreasing* if $a_{i+1} \geq a_i$ for every $i \in I$ and *increasing* if $a_{i+1} > a_i$ for every $i \in I$. We say a sequence $\{a_i\}_{i\in I}$ is *nonincreasing* if $a_{i+1} \leq a_i$ for every $i \in I$ and *decreasing* if $a_{i+1} < a_i$ for every $i \in I$.

Definition A set $A \subset \mathbb{R}$ is said to be *bounded* if there exists a real number M such that $|a| \leq M$ for every $a \in A$. A sequence $\{a_i\}_{i\in I}$ of real numbers is said to be *bounded* if there exists a real number M such that $|a_i| \leq M$ for all $i \in I$.

Theorem If $\{a_i\}_{i\in I}$ is a nondecreasing, bounded sequence of real numbers, then $\{a_i\}_{i\in I}$ converges.

Proof Since $\{a_i\}_{i\in I}$ is bounded, the set of $A = \{a_i : i \in I\}$ has a supremum. Let $L = \sup A$. For any $\epsilon > 0$, there must exist $N \in I$ such that $a_N > L - \epsilon$ (or else $L - \epsilon$ would be an upper bound for A which is smaller than L). But then

$$L - \epsilon < a_N \leq a_i \leq L < L + \epsilon$$

for all $i \geq N$, that is,

$$|a_i - L| < \epsilon$$

for all $i \geq N$. Thus $\{a_i\}_{i\in I}$ converges and

$$L = \lim_{i\to\infty} a_i.$$

We conclude from the previous theorem that every nondecreasing sequence of real numbers either has a limit or is not bounded, that is, is *unbounded*.

Exercise 6.1.1
Show that a nonincreasing, bounded sequence of real numbers must converge.

Definition Let $\{a_i\}_{i\in I}$ be a sequence of real numbers. If for every real number M there exists an integer N such that

$$a_i > M$$

whenever $i > N$, then we say the sequence $\{a_i\}_{i\in I}$ *diverges to positive infinity*, denoted by

$$\lim_{i\to\infty} a_i = +\infty.$$

Similarly, if for every real number M there exists an integer N such that

$$a_i < M$$

whenever $i > N$, then we say the sequence $\{a_i\}_{i \in I}$ *diverges to negative infinity*, denoted by

$$\lim_{i \to \infty} a_i = -\infty.$$

Exercise 6.1.2
Show that a nondecreasing sequence of real numbers either converges or diverges to positive infinity.

Exercise 6.1.3
Show that a nonincreasing sequence of real numbers either converges or diverges to negative infinity.

6.2 Extended real numbers

It is convenient to add the symbols $+\infty$ and $-\infty$ to the real numbers $\mathbb{R}$. Although neither $+\infty$ nor $-\infty$ is a real number, we agree to the following operational conventions:

Given any real number r, $-\infty < r < \infty$.

For any real number r,

$$r + (+\infty) = r - (-\infty) = r + \infty = +\infty,$$

$$r + (-\infty) = r - (+\infty) = r - \infty = -\infty,$$

and

$$\frac{r}{+\infty} = \frac{r}{-\infty} = 0.$$

For any real number $r > 0$, $r \cdot (+\infty) = +\infty$ and $r \cdot (-\infty) = -\infty$. For any real number $r < 0$, $r \cdot (+\infty) = -\infty$ and $r \cdot (-\infty) = +\infty$.

If $a_i = -\infty$, $i = 1, 2, 3, \ldots$, then $\lim_{i \to \infty} a_i = -\infty$; if $a_i = +\infty$, $i = 1, 2, 3, \ldots$, then $\lim_{i \to \infty} a_i = +\infty$.

Note that with the order relation defined in this manner, $+\infty$ is an upper bound and $-\infty$ is a lower bound for any set $A \subset \mathbb{R}$. Thus if $A \subset \mathbb{R}$ does not have a finite upper bound, then $\sup A = +\infty$; similarly, if $A \subset \mathbb{R}$ does not have a finite lower bound, then $\inf A = -\infty$.

When working with extended real numbers, we refer to the elements of $\mathbb{R}$ as *finite* real numbers and the elements $+\infty$ and $-\infty$ as *infinite* real numbers.

Exercise 6.2.1
Do the extended real numbers form a field?

6.3 Limit superior and inferior

Definition Let $\{a_i\}_{i\in I}$ be a sequence of real numbers and, for each $i \in I$, let $u_i = \sup\{a_j : j \geq i\}$. If $u_i = +\infty$ for every $i \in I$, we let

$$\limsup_{i\to\infty} a_i = +\infty;$$

otherwise, we let

$$\limsup_{i\to\infty} a_i = \inf\{u_i : i \in I\}.$$

In either case, we call $\limsup_{n\to\infty} a_n$ the *limit superior* of the sequence $\{a_i\}_{i\in I}$.

Definition Let $\{a_i\}_{i\in I}$ be a sequence of real numbers and, for each $i \in I$, let $l_i = \inf\{a_j : j \geq i\}$. If $l_i = -\infty$ for every $i \in I$, we let

$$\liminf_{i\to\infty} a_i = -\infty;$$

otherwise, we let

$$\liminf_{i\to\infty} a_i = \sup\{l_i : i \in I\}.$$

In either case, we call $\liminf_{n\to\infty} a_n$ the *limit inferior* of the sequence $\{a_i\}_{i\in I}$.

Exercise 6.3.1
Given a sequence $\{a_i\}_{i\in I}$, define $\{u_i\}_{i\in I}$ and $\{l_i\}_{i\in I}$ as in the previous two definitions. Show that

$$\limsup_{i\to\infty} a_i = \lim_{i\to\infty} u_i$$

and

$$\liminf_{i\to\infty} a_i = \lim_{i\to\infty} l_i.$$

Exercise 6.3.2
Find $\limsup_{i\to\infty} a_i$ and $\liminf_{i\to\infty} a_i$ for the sequences $\{a_i\}_{i=1}^{\infty}$ as defined below.
(a) $a_i = (-1)^i$
(b) $a_i = i$
(c) $a_i = 2^{-i}$
(d) $a_i = \dfrac{1}{i}$

The following proposition is often called the *squeeze theorem*.

Proposition Suppose $\{a_i\}_{i\in I}$, $\{b_i\}_{j\in J}$, and $\{c_k\}_{k\in K}$ are sequences of real numbers for which there exists an integer N such that $a_i \leq c_i \leq b_i$ whenever $i > N$. If

$$\lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i,$$

then

$$\lim_{i\to\infty} c_i = \lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i.$$

Proof Let $L = \lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i$. Suppose L is finite. Given $\epsilon > 0$, there exists an integer M such that

$$|a_i - L| < \frac{\epsilon}{4}$$

and

$$|b_i - L| < \frac{\epsilon}{4}$$

whenever $i > M$. Then

$$|a_i - b_i| \le |a_i - L| + |L - b_i| < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}$$

whenever $i > M$. Let K be the larger of N and M. Then

$$|c_i - L| \le |c_i - b_i| + |b_i - L| \le |a_i - b_i| + |b_i - L| < \frac{\epsilon}{2} + \frac{\epsilon}{4} = \frac{3\epsilon}{4} < \epsilon$$

whenever $i > K$. Thus $\lim_{i\to\infty} c_i = L$.

The result when L is infinite is a consequence of the next two exercises.

Exercise 6.3.3
Suppose $\{a_i\}_{i\in I}$ and $\{c_k\}_{k\in K}$ are sequences for which there exists an integer N such that $a_i \le c_i$ whenever $i > N$. Show that if $\lim_{i\to\infty} a_i = +\infty$, then $\lim_{i\to\infty} c_i = +\infty$.

Exercise 6.3.4
Suppose $\{b_j\}_{j\in J}$ and $\{c_k\}_{k\in K}$ are sequences for which there exists an integer N such that $c_i \le b_i$ whenever $i > N$. Show that if $\lim_{i\to\infty} b_i = -\infty$, then $\lim_{i\to\infty} c_i = -\infty$.

Exercise 6.3.5
Suppose $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ are sequences of real numbers with $a_i \le b_i$ for all i larger than some integer N. Assuming both sequences converge, show that

$$\lim_{i\to\infty} a_i \le \lim_{i\to\infty} b_i.$$

Exercise 6.3.6
Show that for any sequence $\{a_i\}_{i\in I}$,

$$\liminf_{i\to\infty} a_i \le \limsup_{i\to\infty} a_i.$$

Proposition Suppose $\{a_i\}_{i\in I}$ is a sequence for which $\limsup_{i\to\infty} a_i = \liminf_{i\to\infty} a_i$. Then

$$\lim_{i\to\infty} a_i = \limsup_{i\to\infty} a_i = \liminf_{i\to\infty} a_i.$$

Proof Let $u_i = \sup\{a_k : k \ge i\}$ and $l_i = \inf\{a_k : k \ge i\}$. Then $l_i \le a_i \le u_i$ for all $i \in I$. Then

$$\lim_{i\to\infty} l_i = \liminf_{i\to\infty} a_i = \limsup_{i\to\infty} a_i = \lim_{i\to\infty} u_i,$$

so the result follows from the squeeze theorem.

Exercise 6.3.7
Suppose u is a real number such that $u \geq 0$ and $u < \epsilon$ for any real number $\epsilon > 0$. Show that $u = 0$.

Definition Suppose $\{a_i\}_{i \in I}$ is a sequence in $\mathbb{R}$. We call $\{a_i\}_{i \in I}$ a *Cauchy sequence* if for every $\epsilon > 0$ there exists an integer N such that

$$|a_i - a_j| < \epsilon$$

whenever both $i > N$ and $j > N$.

Theorem Suppose $\{a_i\}_{i \in I}$ is a Cauchy sequence in $\mathbb{R}$. Then $\{a_i\}_{i \in I}$ converges to a limit $L \in \mathbb{R}$.

Proof Let $u_i = \sup\{a_k : k \geq i\}$ and $l_i = \inf\{a_k : k \geq i\}$. Given any $\epsilon > 0$, there exists $N \in \mathbb{Z}$ such that $|a_i - a_j| < \epsilon$ for all $i, j > N$. Thus, for all $i, j > N$, $a_i < a_j + \epsilon$, and so

$$a_i \leq \inf\{a_j + \epsilon : j \geq i\} = l_i + \epsilon$$

for all $i > N$. Since $\{l_i\}_{i \in I}$ is a nondecreasing sequence,

$$a_i \leq \sup\{l_i + \epsilon : i \in I\} = \liminf_{i \to \infty} a_i + \epsilon$$

for all $i > N$. Hence

$$u_i = \sup\{a_k : k \geq i\} \leq \liminf_{i \to \infty} a_i + \epsilon$$

for all $i > N$. Thus

$$\limsup_{i \to \infty} a_i = \inf\{u_i : i \in I\} \leq \liminf_{i \to \infty} a_i + \epsilon.$$

Since $\liminf_{i \to \infty} a_i \leq \limsup_{i \to \infty} a_i$, it follows that

$$|\limsup_{i \to \infty} a_i - \liminf_{i \to \infty} a_i| \leq \epsilon.$$

Since this is true for every $\epsilon > 0$, we have $\limsup_{i \to \infty} a_i = \liminf_{i \to \infty} a_i$, and so $\{a_i\}_{i \in I}$ converges by the previous proposition.

As a consequence of the previous theorem, we say that $\mathbb{R}$ is a *complete* metric space.

Exercise 6.3.8
Suppose $A \subset \mathbb{R}$, $A \neq \emptyset$, and $s = \sup A$. Show that there exists a sequence $\{a_i\}_{i=1}^{\infty}$ with $a_i \in A$ such that $\lim_{i \to \infty} a_i = s$.

Exercise 6.3.9
Given a real number $x \geq 0$, show that there exists a real number $s \geq 0$ such that $s^2 = x$.

We denote the number s in the previous exercise $\sqrt{x}$.