Lecture 5: More on Real Numbers

5.1 Real numbers: order and metric properties

Given $u \in \mathbb{R}$, we say that u is *positive*, written $u > 0$, if u is the equivalence class of a Cauchy sequence $\{a_i\}_{i \in I}$ for which there exists a rational number $\epsilon > 0$ and an integer N such that $a_i > \epsilon$ for every $i > N$. A real number $u \in \mathbb{R}$ is said to be *negative* if $-u > 0$. We let $\mathbb{R}^+$ denote the set of all postive real numbers.

Exercise 5.1.1
Show that if $u \in \mathbb{R}$, then one and only one of the following is true: (a) $u > 0$, (b) $u < 0$, or (c) $u = 0$.

Exercise 5.1.2
Show that if $a, b \in \mathbb{R}^+$, then $a + b \in \mathbb{R}^+$.

Given real numbers u and v, we say u is *greater* than v, written $u > v$, or, equivalently, v is *less* than u, written, $v < u$, if $u - v > 0$. We write $u \geq v$, or, equivalently, $v \leq u$, to indicate that u is either greater than or equal to v. We say that u is *nonnegative* if $u \geq 0$.

Exercise 5.1.3
Show that $\mathbb{R}$ is an ordered field, that is, verify the following:
(a) For any $a, b \in \mathbb{R}$, one and only one of the following must hold: (a) $a < b$, (b) $a = b$, (c) $a > b$.
(b) If $a, b, c \in \mathbb{R}$ with $a < b$ and $b < c$, then $a < c$.
(c) If $a, b, c \in \mathbb{R}$ with $a < b$, then $a + c < b + c$.
(d) If $a, b \in \mathbb{R}$ with $a > 0$ and $b > 0$, then $ab > 0$.

Exercise 5.1.4
Show that if $a, b \in \mathbb{R}$ with $a > 0$ and $b < 0$, then $ab < 0$.

Exercise 5.1.5
Show that if $a, b, c \in \mathbb{R}$ with $a < b$, then $ac < bc$ if $c > 0$ and $ac > bc$ if $c < 0$.

Exercise 5.1.6
Show that if $a, b \in \mathbb{R}$ with $a < b$, then for any real number λ with $0 < \lambda < 1$, $a < \lambda a + (1 - \lambda)b < b$.

For any $a \in \mathbb{R}$, we call

$$|a| = \begin{cases} a, & \text{if } a \geq 0, \\ -a, & \text{if } a < 0, \end{cases}$$

the *absolute value* of a.

Exercise 5.1.7
Show that for any $a \in \mathbb{R}$, $-|a| \leq a \leq |a|$.

Proposition For any $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.

Proof If $a + b \geq 0$, then

$$|a| + |b| - |a + b| = |a| + |b| - a - b = (|a| - a) + (|b| - b).$$

Both of the terms on the right are nonnegative by Exercise 5.1.7. Hence the sum is nonnegative and the proposition follows. If $a + b < 0$, then

$$|a| + |b| - |a + b| = |a| + |b| + a + b = (|a| + a) + (|b| + b).$$

Again, both of the terms on the right are nonnegative by Exercise 5.1.7. Hence the sum is nonnegative and the proposition follows.

It is now easy to show that the absolute value satisfies

(1) $|a - b| \geq 0$ for all $a, b \in \mathbb{R}$, with $|a - b| = 0$ if and only if $a = b$,
(2) $|a - b| = |b - a|$ for all $a, b \in \mathbb{R}$,
(3) $|a - b| \leq |a - c| + |c - b|$ for all $a, b, c \in \mathbb{R}$.

These properties show that the function

$$d(a, b) = |a - b|$$

is a metric, and we will call $|a - b|$ the *distance* from a to b.

Proposition Given $a \in \mathbb{R}^+$, there exist $r, s \in \mathbb{Q}$ such that $0 < r < a < s$.

Proof Let $\{u\}_{i \in I}$ be a Cauchy sequence in the equivalence class of a. Since $a > 0$, there exists a rational $\epsilon > 0$ and an integer N such that $u_i > \epsilon$ for all $i > N$. Let $r = \frac{\epsilon}{2}$. Then $u_i - r > \frac{\epsilon}{2}$ for every $i > N$, so $a - r > 0$, that is, $0 < r < a$.

Now choose an integer M so that $|u_i - u_j| < 1$ for all $i, j > M$. Let $s = u_{M+1} + 2$. Then

$$s - u_i = u_{M+1} + 2 - u_i > 1$$

for all $i > M$. Hence $s > a$.

Proposition $\mathbb{R}$ is an archimedean ordered field.

Proof Given real numbers a and b with $0 < a < b$, let r and s be rational numbers for which $0 < r < a < b < s$. Since $\mathbb{Q}$ is a an archimedean field, there exists an integer n such that $nr > s$. Hence

$$na > nr > s > b.$$

Proposition Given $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.

Proof Let $\{u\}_{i \in I}$ be a Cauchy sequence in the equivalence class of a and let $\{v\}_{j \in J}$ be in the equivalence class of b. Since $b - a > 0$, there exists a rational $\epsilon > 0$ and an integer N such that $v_i - u_i > \epsilon$ for all $i > N$. Now choose an integer M so that $|u_i - u_j| < \frac{\epsilon}{4}$ for all $i, j > M$. Let $r = u_{M+1} + \frac{\epsilon}{2}$. Then

$$\begin{aligned} r - u_i &= u_{M+1} + \frac{\epsilon}{2} - u_i \\ &= \frac{\epsilon}{2} - (u_i - u_{M+1}) \\ &> \frac{\epsilon}{2} - \frac{\epsilon}{4} \\ &= \frac{\epsilon}{4} \end{aligned}$$

for all $i > M$ and

$$\begin{aligned} v_i - r = v_i - u_{M+1} - \frac{\epsilon}{2} &= (v_i - u_i) - (u_{M+1} - u_i) - \frac{\epsilon}{2} \\ &> \epsilon - \frac{\epsilon}{4} - \frac{\epsilon}{2} \\ &= \frac{\epsilon}{4} \end{aligned}$$

for all i larger than the larger of N and M. Hence $a < r < b$.

5.2 Real numbers: upper bounds

Definition Let $A \subset \mathbb{R}$. If $s \in \mathbb{R}$ is such that $s \geq a$ for every $a \in A$, then we call s an *upper bound* for A. If s is an upper bound for A with the property that $s \leq t$ whenever t is an upper bound for A, then we call s the *supremum*, or *least upper bound*, of A, denoted $s = \sup A$. Similarly, if $r \in \mathbb{R}$ is such that $r \leq a$ for every $a \in A$, then we call r a *lower bound* for A. If r is a lower bound for A with the property that $r \geq t$ whenever t is a lower bound for A, then we call r the *infimum*, or *greatest lower bound*, of A, denoted $r = \inf A$.

Theorem Suppose $A \subset \mathbb{R}$, $A \neq \emptyset$, has an upper bound. Then $\sup A$ exists.

Proof Let $a \in A$ and let b be an upper bound for A. Define sequences $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ as follows: Let $a_1 = a$ and $b_1 = b$. For $i > 1$, let

$$c = \frac{a_{i-1} + b_{i-1}}{2}.$$

If c is an upper bound for A, let $a_i = a_{i-1}$ and let $b_i = c$; otherwise, let $a_i = c$ and $b_i = b_{i-1}$. Then

$$|b_i - a_i| = \frac{|b-a|}{2^{i-1}}$$

for $i = 1, 2, 3, \ldots$. Now, for $i = 1, 2, 3, \ldots$, let r_i be a rational number such that $a_i < r_i < b_i$. Given any $\epsilon > 0$, we may choose N so that

$$2^N > \frac{|b-a|}{\epsilon}.$$

Then, whenever $i > N$ and $j > N$,

$$|r_i - r_j| < |b_{N+1} - a_{N+1}| = \frac{|b-a|}{2^N} < \epsilon.$$

Hence $\{r_i\}_{i=1}^{\infty}$ is a Cauchy sequence. Let $s \in \mathbb{R}$ be the equivalence class of $\{r_i\}_{i=1}^{\infty}$. Note that, for $i = 1, 2, 3, \ldots$, $a_i \leq s \leq b_i$.

Now if s is not an upper bound for A, then there exists $a \in A$ with $a > s$. Let $\delta = a - s$ and choose an integer N such that

$$2^N > \frac{|b-a|}{\delta}.$$

Then

$$b_{N+1} \leq s + \frac{|b-a|}{2^N} < s + \delta = a.$$

But, by consruction, b_{N+1} is an upper bound for A. Thus s must be an upper bound for A.

Now suppose t is another upper bound for A and $t < s$. Let $\delta = s - t$ and choose an integer N such that

$$2^N > \frac{|b-a|}{\delta}.$$

Then

$$a_{N+1} \geq s - \frac{|b-a|}{2^N} > s - \delta = t,$$

which imples that a_{N+1} is an upper bound for A. But, by construction, a_{N+1} is not an upper bound for A. Hence s must be the least upper bound for A, that is, $s = \sup A$.

Exercise 5.2.1

Show that if $A \subset \mathbb{R}$ is nonempty and has an lower bound, then $\inf A$ exists. (Hint: You may wish to first show that $\inf A = -\sup(-A)$, where $-A = \{x : -x \in A\}$.