Lecture 4: Real Numbers

4.1 Real numbers: definition

Let C be the set of all Cauchy sequences of rational numbers. We define a relation on C as follows: if $\{a_i\}_{i \in I}$ and $\{b_j\}_{j \in J}$ are Cauchy sequences in $\mathbb{Q}$, then $\{a_i\}_{i \in I} \sim \{b_j\}_{j \in J}$, which we will write more simply as $a_i \sim b_i$, if for every rational number $\epsilon > 0$, there exists an integer N such that

$$|a_i - b_i| < \epsilon$$

whenever $i > N$. This relation is clearly reflexive and symmetric. To show that it is also transitive, and hence an equivalence relation, suppose $a_i \sim b_i$ and $b_i \sim c_i$. Given $\epsilon > 0$, choose N so that

$$|a_i - b_i| < \frac{\epsilon}{2}$$

for all $i > N$ and M so that

$$|b_i - c_i| < \frac{\epsilon}{2}$$

for all $i > M$. Let L be the larger of N and M. Then, for all $i > L$,

$$|a_i - c_i| \leq |a_i - b_i| + |b_i - c_i| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $a_i \sim c_i$. We call the set of equivalence classes of C the *real numbers*, denoted $\mathbb{R}$. Note that if $a \in \mathbb{Q}$, we may identify a with the equivalence class of the sequence $\{b_i\}_{i=1}^{\infty}$ where $b_i = a$, $i = 1, 2, 3, \ldots$, and thus consider $\mathbb{Q}$ to be a subset of $\mathbb{R}$.

Exercise 4.1.1
Suppose

$$\lim_{i \to \infty} a_i = \lim_{i \to \infty} b_i.$$

Show that $a_i \sim b_i$.

4.2 Real numbers: field properties

Suppose $\{a_i\}_{i \in I}$ and $\{b_j\}_{j \in J}$ are both Cauchy sequences of rational numbers. Let $K = I \cap J$ and define a new sequence $\{s_i\}_{k \in K}$ by setting $s_k = a_k + b_k$. Given any rational $\epsilon > 0$, choose integers N and M such that

$$|a_i - a_j| < \frac{\epsilon}{2}$$

for all $i, j > N$ and

$$|b_i - b_j| < \frac{\epsilon}{2}$$

for all $i, j > M$. If L is the larger of N and M, then, for all $i, j > L$,

$$|s_i - s_j| = |(a_i - a_j) + (b_i - b_j)| \leq |a_i - a_j| + |b_i - b_j| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

showing that $\{s_i\}_{k\in K}$ is also a Cauchy sequence. Moreover, suppose $a_i \sim c_i$ and $b_i \sim d_i$. Given $\epsilon > 0$, choose N so that

$$|a_i - c_i| < \frac{\epsilon}{2}$$

for all $i > N$ and choose M so that

$$|b_i - d_i| < \frac{\epsilon}{2}$$

for all $i > M$. If L is the larger of N and M, then, for all $i > L$,

$$|(a_i + b_i) - (c_i + d_i)| \leq |a_i - c_i| + |b_i - d_i| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $a_i + b_i \sim c_i + d_i$. Thus if $u, v \in \mathbb{R}$, with u being the equivalence class of $\{a_i\}_{i\in I}$ and v being the equivalence class of $\{b_j\}_{j\in J}$, then we may unambiguously define $u + v$ to be the equivalence class of $\{a_i + b_i\}_{i\in K}$, where $K = I \cap J$.

Suppose $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ are both Cauchy sequences of rational numbers. Let $K = I \cap J$ and define a new sequence $\{p_k\}_{k\in K}$ by setting $p_k = a_k b_k$. Let $B > 0$ be an upper bound for the set $\{|a_i| : i \in I\} \cup \{|b_j| : j \in J\}$. Given $\epsilon > 0$, choose integers N and M such that

$$|a_i - a_j| < \frac{\epsilon}{2B}$$

for all $i, j > N$ and

$$|b_i - b_j| < \frac{\epsilon}{2B}$$

for all $i, j > M$. If L is the larger of N and M, then

$$\begin{aligned}
|p_i - p_j| &= |a_i b_i - a_j b_j| \\
&= |a_i b_i - a_j b_i + a_j b_i - a_j b_j| \\
&= |b_i(a_i - a_j) + a_j(b_i - b_j| \\
&\leq |b_i(a_i - a_j| + |a_j(b_i - b_j)| \\
&= |b_i||a_i - a_j| + |a_j||b_i - b_j| \\
&< B\frac{\epsilon}{2B} + B\frac{\epsilon}{2B} = \epsilon.
\end{aligned}$$

Hence $\{p_k\}_{k\in K}$ is a Cauchy sequence. Now suppose $a_i \sim c_i$ and $b_i \sim d_i$. Let $B > 0$ be an upper bound for the set $\{|b_j| : j \in J\} \cup \{|c_i| : i \in H\}$, where H is the appropriate index set. Given $\epsilon > 0$, choose integers N and M such that

$$|a_i - c_i| < \frac{\epsilon}{2B}$$

for all $i > N$ and

$$|b_i - d_i| < \frac{\epsilon}{2B}$$

for all $i > M$. If L is the larger of N and M, then

$$\begin{aligned} |a_i b_i - c_i d_i| &= |a_i b_i - b_i c_i + b_i c_i - c_i d_i| \\ &= |b_i(a_i - c_i) + c_i(b_i - d_i| \\ &\leq |b_i(a_i - c_i| + |c_i(b_i - d_i)| \\ &= |b_i||a_i - c_i| + |c_i||b_i - d_i| \\ &< B\frac{\epsilon}{2B} + B\frac{\epsilon}{2B} = \epsilon. \end{aligned}$$

Hence $a_i b_i \sim c_i d_i$. Thus if $u, v \in \mathbb{R}$, with u being the equivalence class of $\{a_i\}_{i \in I}$ and v being the equivalence class of $\{b_j\}_{j \in J}$, then we may unambiguously define uv to be the equivalence class of $\{a_i b_i\}_{i \in K}$, where $K = I \cap J$.

If $u \in \mathbb{R}$, we define $-u = (-1)u$. Note that if $\{a_i\}_{i \in I}$ is a Cauchy sequence of rational numbers in the equivalence class of u, then $\{-a_i\}_{i \in I}$ is a Cauchy sequence in the equivalence class of $-u$.

We will say that a sequence $\{a_i\}_{i \in I}$ is *bounded away from 0* if there exists a rational number $\delta > 0$ and an integer N such that $|a_i| > \delta$ for all $i > N$. It should be clear that any sequence which converges to 0 is not bounded away from 0. Moreover, as a consequence of the next exercise, any Cauchy sequence which does not converge to 0 must be bounded away from 0.

Exercise 4.2.1

Suppose $\{a_i\}_{i \in I}$ is a Cauchy sequence which is not bounded away from 0. Show that the sequence converges and $\lim_{i \to \infty} a_i = 0$.

Exercise 4.2.2

Suppose $\{a_i\}_{i \in I}$ is a Cauchy sequence which is bounded away from 0 and $a_i \sim b_i$. Show that $\{b_j\}_{j \in J}$ is also bounded away from 0.

Now suppose $\{a_i\}_{i \in I}$ is a Cauchy sequence which is bounded away from 0 and choose $\delta > 0$ and N so that $|a_i| > \delta$ for all $i > N$. Define a new sequence $\{c_i\}_{i=N+1}^{\infty}$ by setting

$$c_i = \frac{1}{a_i}, i = N + 1, N + 2, \ldots.$$

Given $\epsilon > 0$, choose M so that

$$|a_i - a_j| < \epsilon\delta^2$$

for all $i, j > M$. Let L be the larger of N and M. Then, for all $i, j > L$, we have

$$\begin{aligned} |c_i - c_j| &= \left|\frac{1}{a_i} - \frac{1}{a_j}\right| \\ &= \left|\frac{a_j - a_i}{a_i a_j}\right| \\ &= \frac{|a_j - a_i|}{|a_i a_j|} \\ &< \frac{\epsilon\delta^2}{\delta^2} = \epsilon. \end{aligned}$$

Hence $\{c_i\}_{i=N+1}^{\infty}$ is a Cauchy sequence. Now suppose $a_i \sim b_i$. By Exercise 4.2.2 we know that $\{b_j\}_{j\in J}$ is also bounded away from 0, so choose $\gamma > 0$ and K such that $|b_j| > \gamma$ for all $j > K$. Given $\epsilon > 0$, choose P so that

$$|a_i - b_i| < \epsilon\delta\gamma.$$

Let S be the larger of N, K, and P. Then, for all $i, j > S$, we have

$$\begin{aligned}\left|\frac{1}{a_i} - \frac{1}{b_i}\right| &= \left|\frac{b_i - a_i}{a_i b_i}\right| \\ &= \frac{|b_i - a_i|}{|a_i b_i|} \\ &< \frac{\epsilon\delta\gamma}{\delta\gamma} = \epsilon.\end{aligned}$$

Hence $\frac{1}{a_i} \sim \frac{1}{b_i}$. Thus if $u \neq 0$ is a real number which is the equivalence class of $\{a_i\}_{i\in I}$ (necesssarily bounded away from 0), then we may define

$$a^{-1} = \frac{1}{a}$$

to be the equivalence class of

$$\left\{\frac{1}{a_i}\right\}_{i=N+1}^{\infty},$$

where N has been chosen so that $|a_i| > \delta$ for all $i > N$ and some $\delta > 0$.

It follows immediately from these definitions that $\mathbb{R}$ is a field. That is:

(1) $a + b = b + a$ for all $a, b \in \mathbb{R}$;
(2) $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{R}$;
(3) $ab = ba$ for all $a, b \in \mathbb{R}$;
(4) $(ab)c = a(bc)$ for all $a, b, c \in \mathbb{R}$;
(5) $a(b + c) = ab + ac$ for all $a, b, c \in \mathbb{R}$;
(6) $a + 0 = a$ for all $a \in \mathbb{R}$;
(7) $a + (-a) = 0$ for all $a \in \mathbb{R}$;
(8) $1a = a$ for all $a \in \mathbb{R}$;
(9) if $a \in \mathbb{R}$, $a \neq R$, then $aa^{-1} = 1$.