Lecture 3: Sequences of Rational Numbers

3.1 Rational numbers: upper and lower bounds

Definition Let $A \subset \mathbb{Q}$. If $s \in \mathbb{Q}$ is such that $s \geq a$ for every $a \in A$, then we call s an *upper bound* for A. If s is an upper bound for A with the property that $s \leq t$ whenever t is an upper bound for A, then we call s the *supremum*, or *least upper bound*, of A, denoted $s = \sup A$. Similarly, if $r \in \mathbb{Q}$ is such that $r \leq a$ for every $a \in A$, then we call r a *lower bound* for A. If r is a lower bound for A with the property that $r \geq t$ whenever t is a lower bound for A, then we call r the *infimum*, or *greatest lower bound*, of A, denoted $r = \inf A$.

Exercise 3.1.1
Show that the supremum of a set $A \subset \mathbb{Q}$, if it exists, is unique, and thus justify the use of the definite article in the previous definition.

A set which does not have an upper bound will not, *a fortiori*, have a supremum. For example, $\mathbb{Q}$ itself does not have an upper bound. Moreover, even sets which have upper bounds need not have a supremum. Consider the set $A = \{a : a \in \mathbb{Q}, a^2 < 2\}$. Then, for example, 4 is an upper bound for A. Now suppose $s \in \mathbb{Q}$ is the supremum of A. Suppose $s^2 < 2$ and let $\epsilon = 2 - s^2$. By the archimedean property of $\mathbb{Q}$, we may choose $n \in \mathbb{Z}^+$ such that

$$\frac{2s+1}{n} < \epsilon,$$

from which it follows that

$$\frac{2s}{n} + \frac{1}{n^2} = \frac{2s + \frac{1}{n}}{n} \leq \frac{2s+1}{n} < \epsilon.$$

Hence

$$(s + \frac{1}{n})^2 = s^2 + \frac{2s}{n} + \frac{1}{n^2} < s^2 + \epsilon = 2,$$

which implies that $s + \frac{1}{n} \in A$. Since $s < s + \frac{1}{n}$, this contradicts the assumption that s is an upper bound for A. So now suppose $s^2 > 2$. Again let $n \in \mathbb{Z}^+$ and note that

$$(s - \frac{1}{n})^2 = s^2 - \frac{2s}{n} + \frac{1}{n^2}.$$

If we let $\epsilon = s^2 - 2$, then we may choose $n \in \mathbb{Z}^+$ so that

$$\frac{2s}{n} < \epsilon.$$

It follows that

$$(s - \frac{1}{n})^2 > s^2 - \epsilon + \frac{1}{n^2} = 2 + \frac{1}{n^2} > 2.$$

Thus $s - \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n} < s$, contradicting the assumption that $s = \sup A$. Thus we must have $s^2 = 2$. However, this is impossible in light of the following proposition. Hence we must conclude that A does not have a supremum.

Proposition There does not exist a rational number s with the property that $s^2 = 2$.

Proof Suppose there exists $s \in \mathbb{Q}$ such that $s^2 = 2$. Choose $a, b \in \mathbb{Z}^+$ so that a and b are relatively prime (that is, they have no factor other than 1 in common) and $s = \frac{a}{b}$. Then

$$\frac{a^2}{b^2} = 2,$$

so $a^2 = 2b^2$. Thus a^2, and hence a, is an even integer. So there exists $c \in \mathbb{Z}^+$ such that $a = 2c$. Hence

$$a^2 = 4c^2 = 2b^2,$$

from which it follows that $b^2 = 2c$, and so b is also an even integer. But this contradicts the assumption that a and b are relatively prime.

Exercise 3.1.2
Show that there does not exist a rational number s with the property that $s^2 = 3$.

Exercise 3.1.3
Show that there does not exist a rational number s with the property that $s^2 = 6$.

Exercise 3.1.4
As above, let $A = \{a : a \in \mathbb{Q}, a^2 < 2\}$.
(a) Show that if $b \in A$ and $0 < a < b$, then $a \in A$.
(b) Show that if $a > 0$, $a \notin A$, and $b > a$, then $b \notin A$.

3.2 Sequences of rational Numbers

Definition Suppose $n \in \mathbb{Z}$, $I = \{n, n+1, n+2, \ldots\}$, and A is a set. A function $\varphi : I \to A$ is called a *sequence* with values in A.

Frequently, we will define a sequence φ by specifying its values with notation such as, for example, $\{\varphi(i)\}_{i \in I}$, or $\{\varphi(i)\}_{i=n}^{\infty}$. Thus, for example, $\{i^2\}_{i=1}^{\infty}$ denotes the sequence $\varphi : \mathbb{Z}^+ \to \mathbb{Z}$ defined by $\varphi(i) = i^2$. Moreover, it is customary to denote the values of a sequence using subscript notation. Thus if $a_i = \varphi(i)$, $i \in I$, then $\{a_i\}_{i \in I}$ denotes the sequence φ. For example, we may define the sequence of the previous example by writing $a_i = i^2$, $i = 1, 2, 3, \ldots$.

Definition Suppose $\{a_i\}_{i \in I}$ is a sequence with values in $\mathbb{Q}$. We say that $\{a_i\}_{i \in I}$ *converges*, and has *limit* L, $L \in \mathbb{Q}$, if for every $\epsilon > 0$, $\epsilon \in \mathbb{Q}$, there exists $N \in \mathbb{Z}$ such that

$$|a_i - L| < \epsilon$$

whenever $i > N$.

If the sequence $\{a_i\}_{i \in I}$ converges to L, we write

$$\lim_{i \to \infty} a_i = L.$$

For example, clearly

$$\lim_{i \to \infty} \frac{1}{i} = 0$$

since, for any rational number $\epsilon > 0$,

$$|\frac{1}{i} - 0| = \frac{1}{i} < \epsilon$$

for any $i > N$ where N is any integer larger than $\frac{1}{\epsilon}$.

Definition Suppose $\{a_i\}_{i \in I}$ is a sequence with values in $\mathbb{Q}$. We call $\{a_i\}_{i \in I}$ a *Cauchy sequence* if for every $\epsilon > 0$, $\epsilon \in \mathbb{Q}$, there exists $N \in \mathbb{Z}$ such that

$$|a_i - a_k| < \epsilon$$

whenever both $i > N$ and $k > N$.

Proposition If $\{a_i\}_{i \in I}$ converges, then $\{a_i\}_{i \in I}$ is a Cauchy sequence.

Proof Suppose $\lim_{i \to \infty} a_i = L$. Given $\epsilon > 0$, choose an integer N such that

$$|a_i - L| < \frac{\epsilon}{2}$$

for all $i > N$. Then for any $i, k > N$, we have

$$|a_i - a_k| = |(a_i - L) + (a_k - L)| \leq |a_i - L| + |a_k - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $\{a_i\}_{i \in I}$ is a Cauchy sequence.

The proposition shows that every convergent sequence in $\mathbb{Q}$ is a Cauchy sequence, but the converse does not hold. For an example, let

$$f(x) = x^2 - 2$$

and consider the sequence constructed as follows: Begin by setting $a_1 = 1$, $b_1 = 2$, and $x_1 = \frac{3}{2}$. If $f(a_1)f(x_1) < 0$, set

$$x_2 = \frac{a_1 + x_1}{2},$$

$a_2 = a_1$, and $b_2 = x_1$; otherwise, set

$$x_2 = \frac{x_1 + b_1}{2},$$

$a_2 = x_1$, and $b_2 = b_1$. In general, given a_n, x_n, and b_n, if $f(a_n)f(x_n) < 0$, set

$$x_{n+1} = \frac{a_n + x_n}{2},$$

$a_{n+1} = a_n$, and $b_{n+1} = x_n$; otherwise, set

$$x_{n+1} = \frac{x_n + b_n}{2},$$

$a_{n+1} = x_n$, and $b_{n+1} = b_n$. Note that for any positive integer N,

$$a_N < x_i < b_N$$

for all $i > N$. Moreover,

$$|b_N - a_N| = \frac{1}{2^{N-1}},$$

so

$$|x_i - x_k| < \frac{1}{2^{N-1}}$$

for all $i, k > N$. Hence given any $\epsilon > 0$, if we choose an integer N such that $2^{N-1} > \frac{1}{\epsilon}$, then

$$|x_i - x_k| < \frac{1}{2^{N-1}} < \epsilon,$$

showing that $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence. Now suppose $\{x_i\}_{i=1}^{\infty}$ converges to $s \in \mathbb{Q}$. Note that we must have

$$a_i \leq s \leq b_i$$

for all $i \in \mathbb{Z}^+$. If $f(s) < 0$, then, since the set $\{a : a \in \mathbb{Q}, a^2 < 2\}$ does not have a supremum, there exists $t \in \mathbb{Q}$ such that $s < t$ and $f(t) < 0$. If we choose N so that

$$\frac{1}{2^{N-1}} < t - s,$$

then

$$|s - b_N| \leq |a_N - b_N| = \frac{1}{2^{N-1}} < t - s.$$

Hence $b_N < t$, which implies that $f(b_N) < 0$. However, the sequence $\{b_i\}_{i=1}^{\infty}$ was constructed so that $f(b_i) > 0$ for all $i \in \mathbb{Z}^+$. Hence we must have $f(s) > 0$. But if $f(s) > 0$, then there exists $t \in \mathbb{Q}$ such that $t < s$ and $f(t) > 0$. We can then choose N so that $t < a_N$, implying that $f(a_N) > 0$. But the sequence $\{a_i\}_{i=1}^{\infty}$ was constructed so that $f(a_i) < 0$ for all $i \in \mathbb{Z}^+$. Hence we must have $f(s) = 0$, which is not possible since $s \in \mathbb{Q}$. Thus we must conclude that $\{x_i\}_{i=1}^{\infty}$ does not converge.