Lecture 3: Sequences of Rational Numbers

3.1 Rational numbers: upper and lower bounds

**Definition** Let \( A \subseteq \mathbb{Q} \). If \( s \in \mathbb{Q} \) is such that \( s \geq a \) for every \( a \in A \), then we call \( s \) an *upper bound* for \( A \). If \( s \) is an upper bound for \( A \) with the property that \( s \leq t \) whenever \( t \) is an upper bound for \( A \), then we call \( s \) the *supremum*, or *least upper bound*, of \( A \), denoted \( s = \sup A \). Similarly, if \( r \in \mathbb{Q} \) is such that \( r \leq a \) for every \( a \in A \), then we call \( r \) a *lower bound* for \( A \). If \( r \) is a lower bound for \( A \) with the property that \( r \geq t \) whenever \( t \) is a lower bound for \( A \), then we call \( r \) the *infimum*, or *greatest lower bound*, of \( A \), denoted \( r = \inf A \).

**Exercise 3.1.1**
Show that the supremum of a set \( A \subseteq \mathbb{Q} \), if it exists, is unique, and thus justify the use of the definite article in the previous definition.

A set which does not have an upper bound will not, *a fortiori*, have a supremum. For example, \( \mathbb{Q} \) itself does not have an upper bound. Moreover, even sets which have upper bounds need not have a supremum. Consider the set \( A = \{ a : a \in \mathbb{Q}, a^2 < 2 \} \). Then, for example, 4 is an upper bound for \( A \). Now suppose \( s \in \mathbb{Q} \) is the supremum of \( A \). Suppose \( s^2 < 2 \) and let \( \epsilon = 2 - s^2 \). By the archimedean property of \( \mathbb{Q} \), we may choose \( n \in \mathbb{Z}^+ \) such that

\[
\frac{2s + 1}{n} < \epsilon,
\]
from which it follows that

\[
\frac{2s}{n} + \frac{1}{n^2} = \frac{2s + \frac{1}{n}}{n} \leq \frac{2s + 1}{n} < \epsilon.
\]

Hence

\[(s + \frac{1}{n})^2 = s^2 + \frac{2s}{n} + \frac{1}{n^2} < s^2 + \epsilon = 2,
\]
which implies that \( s + \frac{1}{n} \in A \). Since \( s < s + \frac{1}{n} \), this contradicts the assumption that \( s \) is an upper bound for \( A \). So now suppose \( s^2 > 2 \). Again let \( n \in \mathbb{Z}^+ \) and note that

\[(s - \frac{1}{n})^2 = s^2 - \frac{2s}{n} + \frac{1}{n^2} < s^2 - 2 = \frac{1}{n^2},
\]

If we let \( \epsilon = s^2 - 2 \), then we may choose \( n \in \mathbb{Z}^+ \) so that

\[
\frac{2s}{n} < \epsilon.
\]

It follows that

\[(s - \frac{1}{n})^2 > s^2 - \epsilon + \frac{1}{n^2} = 2 + \frac{1}{n^2} > 2.
\]
Thus \( s - \frac{1}{n} \) is an upper bound for \( A \) and \( s - \frac{1}{n} < s \), contradicting the assumption that \( s = \sup A \). Thus we must have \( s^2 = 2 \). However, this is impossible in light of the following proposition. Hence we must conclude that \( A \) does not have a supremum.
Proposition There does not exist a rational number $s$ with the property that $s^2 = 2$.

Proof Suppose there exists $s \in \mathbb{Q}$ such that $s^2 = 2$. Choose $a, b \in \mathbb{Z}^+$ so that $a$ and $b$ are relatively prime (that is, they have no factor other than 1 in common) and $s = \frac{a}{b}$. Then
\[ \frac{a^2}{b^2} = 2, \]
so $a^2 = 2b^2$. Thus $a^2$, and hence $a$, is an even integer. So there exists $c \in \mathbb{Z}^+$ such that $a = 2c$. Hence
\[ a^2 = 4c^2 = 2b^2, \]
from which it follows that $b^2 = 2c$, and so $b$ is also an even integer. But this contradicts the assumption that $a$ and $b$ are relatively prime.

Exercise 3.1.2
Show that there does not exist a rational number $s$ with the property that $s^2 = 3$.

Exercise 3.1.3
Show that there does not exist a rational number $s$ with the property that $s^2 = 6$.

Exercise 3.1.4
As above, let $A = \{a : a \in \mathbb{Q}, a^2 < 2\}$.
(a) Show that if $b \in A$ and $0 < a < b$, then $a \in A$.
(b) Show that if $a > 0, a \notin A$, and $b > a$, then $b \notin A$.

3.2 Sequences of rational Numbers

Definition Suppose $n \in \mathbb{Z}, I = \{n, n+1, n+2, \ldots\}$, and $A$ is a set. A function $\varphi : I \to A$ is called a sequence with values in $A$.

Frequently, we will define a sequence $\varphi$ by specifying its values with notation such as, for example, $\{\varphi(i)\}_{i \in I}$, or $\{\varphi(i)\}_{i=n}^{\infty}$. Thus, for example, $\{i^2\}_{i=1}^{\infty}$ denotes the sequence $\varphi : \mathbb{Z}^+ \to \mathbb{Z}$ defined by $\varphi(i) = i^2$. Moreover, it is customary to denote the values of a sequence using subscript notation. Thus if $a_i = \varphi(i), i \in I$, then $\{a_i\}_{i \in I}$ denotes the sequence $\varphi$. For example, we may define the sequence of the previous example by writing $a_i = i^2, i = 1, 2, 3, \ldots$.

Definition Suppose $\{a_i\}_{i \in I}$ is a sequence with values in $\mathbb{Q}$. We say that $\{a_i\}_{i \in I}$ converges, and has limit $L, L \in \mathbb{Q}$, if for every $\varepsilon > 0, \varepsilon \in \mathbb{Q}$, there exists $N \in \mathbb{Z}$ such that
\[ |a_i - L| < \varepsilon \]
whenever $i > N$.

If the sequence $\{a_i\}_{i \in I}$ converges to $L$, we write
\[ \lim_{i \to \infty} a_i = L. \]

For example, clearly
\[ \lim_{i \to \infty} \frac{1}{i} = 0 \]
since, for any rational number \( \epsilon > 0 \),
\[
\left| \frac{1}{i} - 0 \right| = \frac{1}{i} < \epsilon
\]
for any \( i > N \) where \( N \) is any integer larger than \( \frac{1}{\epsilon} \).

**Definition** Suppose \( \{a_i\}_{i \in I} \) is a sequence with values in \( \mathbb{Q} \). We call \( \{a_i\}_{i \in I} \) a **Cauchy sequence** if for every \( \epsilon > 0, \epsilon \in \mathbb{Q} \), there exists \( N \in \mathbb{Z} \) such that
\[
|a_i - a_k| < \epsilon
\]
whenever both \( i > N \) and \( k > N \).

**Proposition** If \( \{a_i\}_{i \in I} \) converges, then \( \{a_i\}_{i \in I} \) is a Cauchy sequence.

**Proof** Suppose \( \lim_{i \to \infty} a_i = L \). Given \( \epsilon > 0 \), choose an integer \( N \) such that
\[
|a_i - L| < \frac{\epsilon}{2}
\]
for all \( i > N \). Then for any \( i, k > N \), we have
\[
|a_i - a_k| = |(a_i - L) + (a_k - L)| \leq |a_i - L| + |a_k - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
Hence \( \{a_i\}_{i \in I} \) is a Cauchy sequence.

The proposition shows that every convergent sequence in \( \mathbb{Q} \) is a Cauchy sequence, but the converse does not hold. For an example, let
\[
f(x) = x^2 - 2
\]
and consider the sequence constructed as follows: Begin by setting \( a_1 = 1 \), \( b_1 = 2 \), and \( x_1 = \frac{3}{2} \). If \( f(a_1)f(x_1) < 0 \), set
\[
x_2 = \frac{a_1 + x_1}{2},
\]
a_2 = a_1, and \( b_2 = x_1 \); otherwise, set
\[
x_2 = \frac{x_1 + b_1}{2},
\]
a_2 = x_1, and \( b_2 = b_1 \). In general, given \( a_n, x_n \), and \( b_n \), if \( f(a_n)f(x_n) < 0 \), set
\[
x_{n+1} = \frac{a_n + x_n}{2},
\]
a_{n+1} = a_n, and \( b_{n+1} = x_n \); otherwise, set
\[
x_{n+1} = \frac{x_n + b_n}{2},
\]
\[a_{n+1} = x_n, \text{ and } b_{n+1} = b_n. \text{ Note that for any positive integer } N,\]
\[a_N < x_i < b_N\]
for all \(i > N\). Moreover,
\[|b_N - a_N| = \frac{1}{2^{N-1}},\]
so
\[|x_i - x_k| < \frac{1}{2^{N-1}}\]
for all \(i, k > N\). Hence given any \(\varepsilon > 0\), if we choose an integer \(N\) such that \(2^{N-1} > \frac{1}{\varepsilon}\), then
\[|x_i - x_k| < \frac{1}{2^{N-1}} < \varepsilon,\]
showing that \(\{x_i\}_{i=1}^\infty\) is a Cauchy sequence. Now suppose \(\{x_i\}_{i=1}^\infty\) converges to \(s \in \mathbb{Q}\). Note that we must have
\[a_i \leq s \leq b_i\]
for all \(i \in \mathbb{Z}^+\). If \(f(s) < 0\), then, since the set \(\{a : a \in \mathbb{Q}, a^2 < 2\}\) does not have a supremum, there exists \(t \in \mathbb{Q}\) such that \(s < t\) and \(f(t) < 0\). If we choose \(N\) so that
\[\frac{1}{2^{N-1}} < t - s,\]
then
\[|s - b_N| \leq |a_N - b_N| = \frac{1}{2^{N-1}} < t - s.\]
Hence \(b_N < t\), which implies that \(f(b_N) < 0\). However, the sequence \(\{b_i\}_{i=1}^\infty\) was constructed so that \(f(b_i) > 0\) for all \(i \in \mathbb{Z}^+\). Hence we must have \(f(s) > 0\). But if \(f(s) > 0\), then there exists \(t \in \mathbb{Q}\) such that \(t < s\) and \(f(t) > 0\). We can then choose \(N\) so that \(t < a_N\), implying that \(f(a_N) > 0\). But the sequence \(\{a_i\}_{i=1}^\infty\) was constructed so that \(f(a_i) < 0\) for all \(i \in \mathbb{Z}^+\). Hence we must have \(f(s) = 0\), which is not possible since \(s \in \mathbb{Q}\). Thus we must conclude that \(\{x_i\}_{i=1}^\infty\) does not converge.