Lecture 26: The Logarithm and Exponential Functions

26.1 The logarithm function

Definition Given a positive real number x, we call

$$\log(x) = \int_1^x \frac{1}{t}\, dt$$

the *logarithm* of x.

Note that $\log(1) = 0$, $\log(x) < 0$ when $0 < x < 1$, and $\log(x) > 0$ when $x > 1$.

Proposition The function $f(x) = \log(x)$ is an increasing, differentiable function with

$$f'(x) = \frac{1}{x}$$

for all $x > 0$.

Proof Using the fundamental theorem of calculus, we have

$$f'(x) = \frac{1}{x} > 0$$

for all $x > 0$, from which the result follows.

Proposition For any $x > 0$,

$$\log\left(\frac{1}{x}\right) = -\log(x).$$

Proof Using the substitution $t = \frac{1}{u}$, we have

$$\log\left(\frac{1}{x}\right) = \int_1^{\frac{1}{x}} \frac{1}{t}\, dt = \int_1^x u\Big(-\frac{1}{u^2}\Big)\, du = -\int_1^x \frac{1}{u}\, du = -\log(x).$$

Proposition For any positive real numbers x and y,

$$\log(xy) = \log(x) + \log(y).$$

Proof Using the substitution $t = xu$, we have

$$\begin{aligned}
\log(xy) &= \int_1^{xy} \frac{1}{t}\, dt \\
&= \int_{\frac{1}{x}}^{y} \frac{x}{xu}\, du \\
&= \int_{\frac{1}{x}}^{1} \frac{1}{u}\, du + \int_1^y \frac{1}{u}\, du \\
&= -\int_1^{\frac{1}{x}} \frac{1}{u}\, du + \log(y) \\
&= -\log\left(\frac{1}{x}\right) + \log(y) \\
&= \log(x) + \log(y).
\end{aligned}$$

Proposition If $r \in \mathbb{Q}$ and x is a positive real number, then

$$\log(x^r) = r\log(x).$$

Proof Using the substitution $t = u^r$, we have

$$\log(x^r) = \int_1^{x^r} \frac{1}{t}\,dt = \int_1^{x} \frac{ru^{r-1}}{u^r}\,du = r\int_1^{x} \frac{1}{u}\,du = r\log(x).$$

Proposition We have

$$\lim_{x\to+\infty} \log(x) = +\infty$$

and

$$\lim_{x\to 0^+} \log(x) = -\infty.$$

Proof Given a real number M, choose an integer n for which $n\log(2) > M$ (this can be done since $\log(2) > 0$). Then for any $x > 2^n$, we have

$$\log(x) > \log(2^n) = n\log(2) > M.$$

Hence $\lim_{x\to+\infty}\log(x) = +\infty$.

Similarly, given any real number M, we may choose an integer n for which $-n\log(2) < M$. Then for any $0 < x < \frac{1}{2^n}$, we have

$$\log(x) < \log\left(\frac{1}{2^n}\right) = -n\log(2) < M.$$

Hence $\lim_{x\to 0^+}\log(x) = -\infty$.

Note that the logarithm function has domain $(0, +\infty)$ and range $(-\infty, +\infty)$

Exercise 26.1.1
Show that for any rational number $\alpha > 0$,

$$\lim_{x\to+\infty} x^\alpha = +\infty.$$

Proposition For any rational number $\alpha > 0$,

$$\lim_{x\to+\infty} \frac{\log(x)}{x^\alpha} = 0.$$

Proof Choose a rational number β such that $0 < \beta < \alpha$. Now for any $t > 1$,

$$\frac{1}{t} < \frac{1}{t}\,t^\beta = \frac{1}{t^{1-\beta}}.$$

Hence
$$\log(x) = \int_1^x \frac{1}{t}\,dt < \int_1^x \frac{1}{t^{1-\beta}}\,dt = \frac{x^\beta - 1}{\beta} < \frac{x^\beta}{\beta}$$
whenever $x > 1$. Thus
$$0 < \frac{\log(x)}{x^\alpha} < \frac{1}{\beta x^{\alpha-\beta}}$$
for $x > 1$. But
$$\lim_{x \to +\infty} \frac{1}{\beta x^{\alpha-\beta}} = 0,$$
so
$$\lim_{x \to +\infty} \frac{\log(x)}{x^\alpha} = 0.$$

Exercise 26.1.2
Show that
$$\lim_{x \to 0^+} x^\alpha \log(x) = 0$$
for any rational number $\alpha > 0$.

26.2 The exponential function

Definition The inverse of the logarithm function is called the *exponential* function. The value of the exponential function at a real number x is denoted $\exp(x)$.

Proposition The exponential function has domain $\mathbb{R}$ and range $(0, +\infty)$. Moreover, the exponential function is increasing and differentiable on $\mathbb{R}$. If $f(x) = \exp(x)$, then $f'(x) = \exp(x)$.

Proof Only the final statement of the proposition requires proof. If $g(x) = \log(x)$, then
$$f'(x) = \frac{1}{g'(\exp(x))} = \exp(x).$$

Proposition For any real numbers x and y,
$$\exp(x + y) = \exp(x)\exp(y).$$

Proof The result follows from
$$\log(\exp(x)\exp(y)) = \log(\exp(x)) + \log(\exp(y)) = x + y.$$

Proposition For any real number x,
$$\exp(-x) = \frac{1}{\exp(x)}.$$

Proof The result follows from

$$\log\left(\frac{1}{\exp(x)}\right) = -\log(\exp(x)) = -x.$$

Exercise 26.2.1
Use Taylor's theorem to show that

$$\exp(1) = e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Proposition For any rational number α,

$$\exp(\alpha) = e^{\alpha}.$$

Proof Since $\log(e) = 1$, we have

$$\log(e^{\alpha}) = \alpha \log(e) = \alpha.$$

Definition If α is an irrational number, we define

$$e^{\alpha} = \exp(\alpha).$$

Note that for any real numbers x and y,

$$e^{x+y} = e^x e^y$$

and

$$e^{-x} = \frac{1}{e^x}.$$

Moreover, $\log(e^x) = x$ and, if $x > 0$, $e^{\log(x)} = x$.

Definition If x and a are real numbers with $a > 0$, we define

$$a^x = e^{x \log(a)}.$$

Exercise 26.2.2
Suppose $f : (0, +\infty) \to \mathbb{R}$ is given by $f(x) = x^a$, where $a \in \mathbb{R}$, $a \neq 0$. Show that $f'(x) = ax^{a-1}$.

Exercise 26.2.3
Suppose a is a positive real number and $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = a^x$. Show that $f'(x) = a^x \log(a)$.

Proposition For any real number $\alpha > 0$,

$$\lim_{x \to +\infty} x^{\alpha} e^{-x} = 0.$$

Proof We know that

$$\lim_{y \to +\infty} \frac{\log(y)}{y^{\frac{1}{\alpha}}} = 0.$$

Hence

$$\lim_{y \to +\infty} \frac{(\log(y))^{\alpha}}{y} = 0.$$

Letting $y = e^x$, we have

$$\lim_{x \to +\infty} \frac{x^{\alpha}}{e^x} = 0.$$

Proposition For any real number α,

$$\lim_{x \to +\infty} \left(1 + \frac{\alpha}{x}\right)^x = e^{\alpha}.$$

Proof First note that, letting $x = \frac{1}{h}$,

$$\lim_{x \to +\infty} \left(1 + \frac{\alpha}{x}\right)^x = \lim_{h \to 0^+} (1 + \alpha h)^{\frac{1}{h}} = \lim_{h \to 0^+} e^{\log\left((1+\alpha h)^{\frac{1}{h}}\right)}.$$

Using l'Hôpital's rule, we have

$$\lim_{h \to 0^+} \log\left((1 + \alpha h)^{\frac{1}{h}}\right) = \lim_{h \to 0^+} \frac{\log(1 + \alpha h)}{h} = \lim_{h \to 0^+} \frac{\alpha}{1 + \alpha h} = \alpha,$$

and the result follows from the continuity of the exponential function.

Exercise 26.2.4

The *hyperbolic sine* and *hyperbolic cosine* functions are defined by

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

and

$$\cosh(x) = \frac{e^x + e^{-x}}{2},$$

respectively. Verify the following:

(a) For any real numbers x and y,

$$\sinh(x + y) = \sinh(x)\cosh(y) + \sinh(y)\cosh(x)$$

and

$$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y).$$

(b) For any real number x,

$$\cosh^2(x) - \sinh^2(x) = 1.$$

(c) If $f(x) = \sinh(x)$ and $g(x) = \cosh(x)$, then

$$f'(x) = \cosh(x)$$

and

$$g'(x) = \sinh(x).$$