Lecture 25: The Sine and Cosine Functions

25.1 Definitions

We begin by defining functions

$$s : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$$

and

$$c : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$$

by

$$s(x) = \begin{cases} \dfrac{\tan(x)}{\sqrt{1+\tan^2(x)}}, & \text{if } x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \\ 1, & \text{if } x = \dfrac{\pi}{2} \end{cases}$$

and

$$c(x) = \begin{cases} \dfrac{1}{\sqrt{1+\tan^2(x)}}, & \text{if } x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \\ 0, & \text{if } x = \dfrac{\pi}{2}. \end{cases}$$

Note that

$$\lim_{x \to \frac{\pi}{2}^-} s(x) = \lim_{y \to +\infty} \frac{y}{\sqrt{1+y^2}} = \lim_{y \to +\infty} \frac{1}{\sqrt{1+\frac{1}{y^2}}} = 1$$

and

$$\lim_{x \to \frac{\pi}{2}^-} c(x) = \lim_{y \to +\infty} \frac{1}{\sqrt{1+y^2}} = \lim_{y \to +\infty} \frac{\frac{1}{y}}{\sqrt{1+\frac{1}{y^2}}} = 0,$$

which shows that both s and c are continuous functions.

Next, we extend the definitions of s and c to functions

$$S : \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right] \to \mathbb{R}$$

and

$$C : \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right] \to \mathbb{R}$$

by defining

$$S(x) = \begin{cases} s(x), & \text{if } x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right], \\ -s(x-\pi), & \text{if } x \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right] \end{cases}$$

and

$$C(x) = \begin{cases} c(x), & \text{if } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right], \\ -c(x-\pi), & \text{if } x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right]. \end{cases}$$

Note that

$$\lim_{x \to \frac{\pi}{2}^+} S(x) = \lim_{x \to -\frac{\pi}{2}^+} -s(x) = -\lim_{y \to -\infty} \frac{y}{\sqrt{1+y^2}} = -\lim_{y \to -\infty} \frac{1}{-\sqrt{1+\frac{1}{y^2}}} = 1$$

and

$$\lim_{x \to \frac{\pi}{2}^+} C(x) = \lim_{x \to -\frac{\pi}{2}^+} -c(x) = -\lim_{y \to -\infty} \frac{1}{\sqrt{1+y^2}} = -\lim_{y \to -\infty} \frac{\frac{1}{y}}{-\sqrt{1+\frac{1}{y^2}}} = 0,$$

which shows that both S and C are continuous at $\frac{\pi}{2}$. Thus both S and C are continuous.

Finally, for any $x \in \mathbb{R}$, let

$$g(x) = \sup\{n : n \in \mathbb{Z}, -\frac{\pi}{2} + 2n\pi < x\}.$$

Definition With the notation as above, for any $x \in \mathbb{R}$ we call

$$\sin(x) = S(x - 2\pi g(x))$$

and

$$\cos(x) = C(x - 2\pi g(x))$$

the *sine* and *cosine* of x, respectively.

Proposition The sine and cosine functions are continuous on $\mathbb{R}$.

Proof From the definitions, it is sufficient to verify continuity at $\frac{3\pi}{2}$. Now

$$\lim_{x \to \frac{3\pi}{2}^-} \sin(x) = \lim_{x \to \frac{3\pi}{2}^-} S(x) = S\left(\frac{3\pi}{2}\right) = -s\left(\frac{\pi}{2}\right) = -1$$

and

$$\begin{aligned} \lim_{x \to \frac{3\pi}{2}^+} \sin(x) &= \lim_{x \to \frac{3\pi}{2}^+} S(x - 2\pi) \\ &= \lim_{x \to -\frac{\pi}{2}^+} s(x) \\ &= \lim_{y \to -\infty} \frac{y}{\sqrt{1+y^2}} \\ &= \lim_{y \to -\infty} \frac{1}{-\sqrt{1+\frac{1}{y^2}}} \\ &= -1, \end{aligned}$$

and so sine is continuous at $\frac{3\pi}{2}$. Similarly,

$$\lim_{x \to \frac{3\pi}{2}^-} \cos(x) = \lim_{x \to \frac{3\pi}{2}^-} C(x) = C\left(\frac{3\pi}{2}\right) = -c\left(\frac{\pi}{2}\right) = 0$$

and

$$\begin{aligned}
\lim_{x \to \frac{3\pi}{2}^+} \cos(x) &= \lim_{x \to \frac{3\pi}{2}^+} C(x - 2\pi) \\
&= \lim_{x \to -\frac{\pi}{2}^+} c(x) \\
&= \lim_{y \to -\infty} \frac{1}{\sqrt{1+y^2}} \\
&= \lim_{y \to -\infty} \frac{\frac{1}{y}}{-\sqrt{1+\frac{1}{y^2}}} \\
&= 0,
\end{aligned}$$

and so cosine is continuous at $\frac{3\pi}{2}$.

25.2 Properties of sine and cosine

Proposition The sine and cosine functions are periodic with period 2π.

Proof The result follows immediately from the definitions.

Proposition For any $x \in \mathbb{R}$, $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$.

Proof The result follows immediately from the definitions.

Proposition For any $x \in \mathbb{R}$, $\sin^2(x) + \cos^2(x) = 1$.

Proof The result follows immediately from the definition of s and c.

Proposition The range of both the sine and cosine functions is $[-1, 1]$.

Proof The result follows immediately from the definitions along with the facts that

$$\sqrt{1+y^2} \geq \sqrt{y^2} = |y|$$

and

$$\sqrt{1+y^2} \geq 1$$

for any $y \in \mathbb{R}$.

Proposition For any x in the domain of the tangent function,

$$\tan(x) = \frac{\sin(x)}{\cos(x)}.$$

Proof The result follows immediately from the definitions.

Proposition For any x in the domain of the tangent function,

$$\sin^2(x) = \frac{\tan^2(x)}{1+\tan^2(x)}$$

and

$$\cos^2(x) = \frac{1}{1+\tan^2(x)}.$$

Proof The result follows immediately from the definitions.

Proposition For any $x, y \in \mathbb{R}$,

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

Proof First suppose x, y, and $x+y$ are in the domain of the tangent function. Then

$$\begin{aligned}
\cos^2(x+y) &= \frac{1}{1+\tan^2(x+y)} \\
&= \frac{1}{1+\left(\dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}\right)^2} \\
&= \frac{(1-\tan(x)\tan(y))^2}{(1-\tan(x)\tan(y))^2+(\tan(x)+\tan(y))^2} \\
&= \frac{(1-\tan(x)\tan(y))^2}{(1+\tan^2(x))(1+\tan^2(y))} \\
&= \left(\frac{1}{\sqrt{1+\tan^2(x)}\sqrt{1+\tan^2(y)}} - \frac{\tan(x)\tan(y)}{\sqrt{1+\tan^2(x)}\sqrt{1+\tan^2(y)}}\right)^2 \\
&= (\cos(x)\cos(y)-\sin(x)\sin(y))^2.
\end{aligned}$$

Hence

$$\cos(x+y) = \pm(\cos(x)\cos(y)-\sin(x)\sin(y)).$$

Consider a fixed value of x. Note that the positive sign must be chosen when $y = 0$. Moreover, increasing y by π changes the sign on both sides, so the positive sign must be chosen when y is any multiple of π. Since sine and cosine are continuous functions,the choice of sign could change only at points at which both sides are 0, but these points are separated by a distance of π, so we must always choose the positive sign. Hence we have

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

for all $x, y \in \mathbb{R}$ for which x, y, and $x+y$ are in the domain of the tangent function. The identity for the other values of x and y now follows from the continuity of the sine and cosine functions.

Proposition For any $x, y \in \mathbb{R}$,

$$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x).$$

Exercise 25.2.1
Prove the previous proposition.

Exercise 25.2.2
Show that for any $x \in \mathbb{R}$,

$$\sin\left(\frac{\pi}{2} - x\right) = \cos(x)$$

and

$$\cos\left(\frac{\pi}{2} - x\right) = \sin(x).$$

Exercise 25.2.3
Show that for any $x \in \mathbb{R}$,

$$\sin(2x) = 2\sin(x)\cos(x)$$

and

$$\cos(2x) = \cos^2(x) - \sin^2(x).$$

Exercise 25.2.4
Show that for any $x \in \mathbb{R}$,

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

and

$$\cos^2(x) = \frac{1+\cos(2x)}{2}.$$

Exercise 25.2.5
Show that

$$\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}},$$

$$\sin\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2},$$

and

$$\sin\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}.$$

25.3 The calculus of the trigonometric functions

Proposition $\lim_{x \to 0} \frac{\arctan(x)}{x} = 1.$

Proof Using l'Hôpital's rule,

$$\lim_{x \to 0} \frac{\arctan(x)}{x} = \lim_{x \to 0} \frac{\frac{1}{1+x^2}}{1} = 1.$$

Proposition $\lim_{x \to 0} \frac{\tan(x)}{x} = 1.$

Proof Letting $x = \arctan(u)$, we have

$$\lim_{x \to 0} \frac{\tan(x)}{x} = \lim_{u \to 0} \frac{u}{\arctan(u)} = 1.$$

Proposition $\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$

Proof We have

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\tan(x)}{x} \cos(x) = 1.$$

Proposition $\lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0.$

Proof We have

$$\begin{aligned}
\lim_{x \to 0} \frac{1 - \cos(x)}{x} &= \lim_{x \to 0} \left(\frac{1 - \cos(x)}{x}\right)\left(\frac{1 + \cos(x)}{1 + \cos(x)}\right) \\
&= \lim_{x \to 0} \frac{1 - \cos^2(x)}{x(1 + \cos(x))} \\
&= \lim_{x \to 0} \left(\frac{\sin(x)}{x}\right)\left(\frac{\sin(x)}{1 + \cos(x)}\right) \\
&= (1)(0) = 0.
\end{aligned}$$

Proposition If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$.

Proof We have

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h} \\
&= \sin(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h} \\
&= \cos(x).
\end{aligned}$$

Proposition If $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$.

Exercise 25.3.1
Prove the previous proposition.

Definition For appropriate $x \in \mathbb{R}$,

$$\cot(x) = \frac{\cos x}{\sin(x)},$$

$$\sec(x) = \frac{1}{\cos(x)},$$

and

$$\csc(x) = \frac{1}{\sin(x)}$$

are called the *cotangent*, *secant*, and *cosecant* of x, respectively.

Exercise 25.3.2
If $f(x) = \tan(x)$, $g(x) = \cot(x)$, $h(x) = \sec(x)$, and $k(x) = \csc(x)$, show that

$$f'(x) = \sec^2(x),$$

$$g'(x) = -\csc^2(x),$$

$$h'(x) = \sec(x)\tan(x),$$

and

$$k'(x) = -\csc(x)\cot(x).$$

Proposition $2\int_{-1}^{1} \sqrt{1-x^2}\,dx = \pi.$

Proof Let $x = \sin(u)$. Then as u varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, x varies from -1 to 1. And, for these values, we have

$$\sqrt{1-x^2} = \sqrt{1-\sin^2(u)} = \sqrt{\cos^2(u)} = |\cos(u)| = \cos(u).$$

Hence

$$\begin{aligned}
\int_{-1}^{1} \sqrt{1-x^2}\,dx = \pi &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(u)du \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos(2u)}{2}\, du \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\, du + \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2u)du \\
&= \frac{\pi}{2} + \frac{1}{4}(\sin(\pi) - \sin(-\pi)) \\
&= \frac{\pi}{2}.
\end{aligned}$$

Exercise 25.3.3

Find the Taylor polynomial P_9 of order 9 for $f(x) = \sin(x)$ at 0. Note that this is equal to the Taylor polynomial of order 10 for f at 0. Is $P_9\left(\frac{1}{2}\right)$ an overestimate or an underestimate for $\sin\left(\frac{1}{2}\right)$? Find an upper bound for the error in this approximation.