Lecture 24: The Tangent Function

24.1 An improper integral

Definition If f is integrable on $[a, b]$ for all $b > a$ and

$$\lim_{b \to +\infty} \int_a^b f(x)dx$$

exists, then we define

$$\int_a^{+\infty} f(x)dx = \lim_{b \to +\infty} \int_a^b f(x)dx.$$

If f is integrable on $[a, b]$ for all $a < b$ and

$$\lim_{a \to -\infty} \int_a^b f(x)dx$$

exists, then we define

$$\int_{-\infty}^b f(x)dx = \lim_{a \to -\infty} \int_a^b f(x)dx.$$

Proposition Suppose f is continuous on $[a, \infty)$ and $f(x) \geq 0$ for all $x \geq a$. If there exists $g : [a, +\infty) \to \mathbb{R}$ for which $\int_a^{+\infty} g(x)dx$ exists and $g(x) \geq f(x)$ for all $x \geq a$, then $\int_a^{+\infty} f(x)dx$ exists.

Proof See Exercise 23.2.5.

Example Suppose

$$f(x) = \frac{1}{1 + x^2}$$

and

$$g(x) = \begin{cases} 1, & \text{if } 0 \leq x < 1, \\ \dfrac{1}{x^2}, & \text{if } x \geq 1. \end{cases}$$

Then, for $b > 1$,

$$\int_0^b g(x)dx = \int_0^1 dx + \int_1^b \frac{1}{x^2}dx = 1 + 1 - \frac{1}{b} = 2 - \frac{1}{b},$$

so

$$\int_0^{+\infty} g(x)dx = \lim_{b \to \infty} \left(2 - \frac{1}{b}\right) = 2.$$

Since $0 < f(x) \leq g(x)$ for all $x \geq 0$, it follows that

$$\int_0^{+\infty} \frac{1}{1 + x^2}$$

exists, and, moreover,

$$\int_0^{+\infty} \frac{1}{1+x^2}\, dx < 2.$$

Also, the substitution $u = -x$ shows that

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\, dx = -\int_{+\infty}^{0} \frac{1}{1+u^2}\, du = \int_0^{+\infty} \frac{1}{1+u^2}\, du.$$

24.2 The arctangent function

Definition For any $x \in \mathbb{R}$, we call

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}\, dt$$

the *arctangent* of x.

Proposition The arctangent function is differentiable at every $x \in \mathbb{R}$. Moreover, if $f(x) = \arctan(x)$, then

$$f'(x) = \frac{1}{1+x^2}.$$

Proof The result follows immediately from the the fundamental theorem of calculus.

Proposition The arctangent is increasing on $\mathbb{R}$.

Proof The result follows immediately from the previous proposition and the fact that

$$\frac{1}{1+x^2} > 0$$

for every $x \in \mathbb{R}$.

Definition $\pi = 2 \lim_{x \to +\infty} \arctan(x) = 2 \int_0^{+\infty} \frac{1}{1+t^2}\, dt.$

The following proposition says that the arctangent function is an odd function.

Proposition For any $x \in \mathbb{R}$, $\arctan(x) = -\arctan(-x)$.

Proof Using the substitution $t = -u$, we have

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}\, dt = -\int_0^{-x} \frac{1}{1+u^2}\, du = -\arctan(-x).$$

It now follows that

$$\lim_{x \to -\infty} \arctan(x) = -\lim_{x \to -\infty} \arctan(-x) = -\frac{\pi}{2}.$$

Hence the range of the arctangent function is

$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Proposition If $x > 0$, then

$$\arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}.$$

Proof Using the substitution $t = \frac{1}{u}$, we have

$$\begin{aligned}\int_0^{\frac{1}{x}} \frac{1}{1+t^2}\, dt &= \int_{+\infty}^{x} \frac{1}{1+\frac{1}{u^2}}\left(-\frac{1}{u^2}\right) du \\ &= -\int_{+\infty}^{x} \frac{1}{1+u^2}\, du \\ &= \int_{x}^{+\infty} \frac{1}{1+u^2}\, du \\ &= \frac{\pi}{2} - \int_0^{x} \frac{1}{1+u^2}\, du \\ &= \frac{\pi}{2} - \arctan(x).\end{aligned}$$

Proposition If $x < 0$, then

$$\arctan(x) + \arctan\left(\frac{1}{x}\right) = -\frac{\pi}{2}.$$

Proof The result follows immediately from the preceding proposition and the fact that arctangent is an odd function.

24.3 The tangent function

Let

$$A = \left\{\frac{\pi}{2} + n\pi : n \in \mathbb{Z}\right\}$$

and $D = \mathbb{R} \setminus A$. Let

$$t : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$$

be the inverse of the arctangent function. Note that t is increasing and differentiable on $(-\frac{\pi}{2}, \frac{\pi}{2})$. We may extend t to a function on D as follows: For any $x \in D$, let

$$g(x) = \sup\left\{n : n \in \mathbb{Z}, -\frac{\pi}{2} + n\pi < x\right\}$$

and define $T(x) = t(x - g(x)\pi)$.

Definition With the notation of the above discussion, for any $x \in D$, the value $T(x)$ is called the *tangent* of x, denoted $\tan(x)$.

Proposition The tangent function has domain D (as defined above), range $\mathbb{R}$, and is differentiable at every point $x \in D$. Moreover, the tangent function is increasing on each interval of the form

$$\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right),$$

$n \in \mathbb{Z}$, with

$$\tan((\frac{\pi}{2} + n\pi)+) = -\infty$$

and

$$\tan((\frac{\pi}{2} + n\pi)-) = +\infty.$$

Definition Let $D \subset \mathbb{R}$. A function $f : D \to \mathbb{R}$ is said to be *periodic* with *period* p if $f(x + p) = f(x)$ for all $x \in D$.

Proposition The tangent function has period π.

Proof The result follows immediately from our definitions.

24.4 The addition formula for tangent

We will now derive the addition formula for tangent. To begin, suppose $y_1, y_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ with $y_1 + y_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Let $x_1 = \tan(y_1)$ and $x_2 = \tan(y_2)$. Note that if $x_1 > 0$, then $x_1 x_2 \geq 1$ would imply that

$$x_2 \geq \frac{1}{x_1},$$

which in turn implies that

$$\arctan(x_1) + \arctan(x_2) \geq \arctan(x_1) + \arctan\left(\frac{1}{x_1}\right) = \frac{\pi}{2}.$$

Hence we would have $y_1 + y_2 \geq \frac{\pi}{2}$, contrary to our assumptions. Similarly, if $x_1 < 0$, then $x_1 x_2 \geq 1$ would imply that

$$x_2 \leq \frac{1}{x_1},$$

which in turn implies that

$$\arctan(x_1) + \arctan(x_2) \leq \arctan(x_1) + \arctan\left(\frac{1}{x_1}\right) = -\frac{\pi}{2}.$$

Hence we would have $y_1 + y_2 \leq \frac{\pi}{2}$, again contrary to our assumptions. Thus we must have $x_1 x_2 < 1$. Moreover, suppose u is a number between $-x_1$ and x_2. If $x_1 > 0$, then

$$x_2 < \frac{1}{x_1},$$

so

$$u < \frac{1}{x_1}.$$

If $x_1 < 0$, then

$$x_2 > \frac{1}{x_1},$$

so

$$u > \frac{1}{x_1}.$$

Now let

$$x = \frac{x_1 + x_2}{1 - x_1 x_2}.$$

We want to show that

$$\arctan(x) = \arctan(x_1) + \arctan(x_2),$$

which will imply that

$$\frac{\tan(y_1) + \tan(y_2)}{1 - \tan(y_1)\tan(y_2)} = \tan(y_1 + y_2).$$

We need to compute

$$\arctan(x) = \arctan\left(\frac{x_1 + x_2}{1 - x_1 x_2}\right) = \int_0^{\frac{x_1+x_2}{1-x_1x_2}} \frac{1}{1+t^2}\, dt.$$

Let

$$t = \varphi(u) = \frac{x_1 + u}{1 - x_1 u},$$

where u varies between $-x_1$, where $t = 0$, and x_2, where $t = x$. Now

$$\varphi'(u) = \frac{(1 - x_1 u) - (x_1 + u)(-x_1)}{(1 - x_1 u)^2} = \frac{1 + x_1^2}{(1 - x_1 u)^2},$$

which is always positive, thus showing that φ is an increasing function, and

$$\begin{aligned}
\frac{1}{1+t^2} &= \frac{1}{1 + \left(\frac{x_1+u}{1-x_1u}\right)^2} \\
&= \frac{(1 - x_1 u)^2}{(1 - x_1 u)^2 + (x_1 + u)^2} \\
&= \frac{(1 - x_1 u)^2}{(1 + x_1^2)(1 + u^2)}.
\end{aligned}$$

Hence

$$\begin{aligned}\arctan(x) &= \int_{-x_1}^{x_2} \frac{1}{1+u^2}\,du \\ &= \int_{-x_1}^{0} \frac{1}{1+u^2}\,du + \int_{0}^{x_2} \frac{1}{1+u^2}\,du \\ &= -\int_{0}^{-x_1} \frac{1}{1+u^2}\,du + \arctan(x_2) \\ &= -\arctan(-x_1) + \arctan(x_2) \\ &= \arctan(x_1) + \arctan(x_2).\end{aligned}$$

Now suppose $y_1 + y_2 > \frac{\pi}{2}$. Then $x_1 > 0$, $x_2 > 0$, and

$$x_2 > \frac{1}{x_1}.$$

Note then that as u increases from $-x_1$ to $\frac{1}{x_1}$, t increases from 0 to $+\infty$; and as u increases from $\frac{1}{x_1}$ to x_2, t increases from $-\infty$ to x. Hence we have

$$\begin{aligned}\arctan(x) + \pi &= \int_{0}^{x} \frac{1}{1+t^2}\,dt + \int_{-\infty}^{0} \frac{1}{1+t^2}\,dt + \int_{0}^{+\infty} \frac{1}{1+t^2}\,dt \\ &= \int_{-\infty}^{x} \frac{1}{1+t^2}\,dt + \int_{0}^{+\infty} \frac{1}{1+t^2}\,dt \\ &= \int_{\frac{1}{x_1}}^{x_2} \frac{1}{1+u^2}\,du + \int_{-x_1}^{\frac{1}{x_1}} \frac{1}{1+u^2}\,du \\ &= \int_{-x_1}^{x_2} \frac{1}{1+u^2}\,du \\ &= \arctan(x_2) - \arctan(-x_1) \\ &= \arctan(x_2) + \arctan(x_1).\end{aligned}$$

Hence

$$\begin{aligned}\tan(y_1 + y_2) = \tan(y_1 + y_2 - \pi) &= \tan(\arctan(x)) \\ &= \frac{x_1 + x_2}{1 - x_1 x_2} \\ &= \frac{\tan(y_1) + \tan(y_2)}{1 - \tan(y_1)\tan(y_2)}.\end{aligned}$$

The case when $x_1 < 0$ may be handled similarly; it then follows that the addition formula holds for all $y_1, y_2 \in (-\frac{\pi}{2}, -\frac{\pi}{2})$. The case for arbitrary $y_1, y_2 \in D$ with $y_1 + y_2 \in D$ then follows from the periodicity of the tangent function.

Proposition For any $x, y \in D$ with $x + y \in D$,

$$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}.$$