# Lecture 23: The Fundamental Theorem of Calculus

## 23.1 The fundamental theorem of calculus

The following result is called the *fundamental theorem of calculus.*

**Theorem** Suppose $f$ is integrable on $[a, b]$ and $F$ is continuous on $[a, b]$ and differentiable on $(a, b)$ with $F'(x) = f(x)$ for all $x \in (a, b)$. Then

$$\int_a^b f = F(b) - F(a).$$

**Proof** Given $\epsilon > 0$, let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ for which $U(f, P) - L(f, P) < \epsilon$. For $i = 1, 2, \ldots, n$, let $t_i \in (x_{i-1}, x_i)$ be points for which

$$F(x_i) - F(x_{i-1}) = f(t_i)(x_i - x_{i-1}).$$

Then

$$\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^{n}(F(x_i) - F(x_{i-1})) = F(b) - F(a).$$

But

$$L(f, P) \le \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \le U(f, P),$$

so

$$\left| F(b) - F(a) - \int_a^b f \right| < \epsilon.$$

Since $\epsilon$ was arbitrary, we conclude that

$$\int_a^b f = F(b) - F(a).$$

The following result is known as *integration by parts.*

**Proposition** Suppose $f$ and $g$ are integrable on $[a, b]$ and $F$ and $G$ are continuous on $[a, b]$ and differentiable on $(a, b)$ with $F'(x) = f(x)$ and $G'(x) = g(x)$ for all $x \in (a, b)$. Then

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx.$$

**Proof** By the fundamental theorem of calculus,

$$\int_a^b (F(x)g(x) + f(x)G(x))dx = F(b)G(b) - F(a)G(a).$$

## 23.2 The other fundamental theorem of calculus

**Proposition** Suppose $f$ is integrable on $[a,b]$ and $F:[a,b]\to\mathbb{R}$ is defined by

$$F(x)=\int_a^x f(t)dt.$$

Then $F$ is uniformly continuous on $[a,b]$.

**Proof** Let $\epsilon>0$ be given and let $M>0$ be such that $|f(x)|\le M$ for all $x\in[a,b]$. Then for any $x,y\in[a,b]$ with $x<y$ and $y-x<\frac{\epsilon}{M}$,

$$|F(y)-F(x)|=\left|\int_x^y f(t)dt\right|\le M(y-x)<\epsilon.$$

Hence $F$ is uniformly continuous on $[a,b]$.

The following theorem is also sometimes called the *fundamental theorem of calculus.*

**Theorem** Suppose $f$ is integrable on $[a,b]$ and continuous at $u\in(a,b)$. If $F:[a,b]\to\mathbb{R}$ is defined by

$$F(x)=\int_a^x f(t)dt,$$

then $F$ is differentiable at $u$ and $F'(u)=f(u)$.

**Proof** Let $\epsilon>0$ be given and choose $\delta>0$ such that $|f(x)-f(u)|<\epsilon$ whenever $|x-u|<\delta$. Then if $0<h<\delta$, we have

$$\left|\frac{F(u+h)-F(u)}{h}-f(u)\right|=\left|\frac{1}{h}\int_u^{u+h} f(t)dt-f(u)\right|=\left|\frac{1}{h}\int_u^{u+h}(f(t)-f(u))dt\right|<\epsilon.$$

If $-\delta<h<0$, then

$$\left|\frac{F(u+h)-F(u)}{h}-f(u)\right|=\left|\frac{1}{h}\int_{u+h}^{u} f(t)dt-f(u)\right|=\left|\frac{1}{h}\int_{u+h}^{u}(f(t)-f(u))dt\right|<\epsilon.$$

Hence

$$F'(u)=\lim_{h\to 0}\frac{F(u+h)-F(u)}{h}=f(u).$$

**Proposition** If $f$ is continuous on $[a,b]$, then there exists a function $F:[a,b]\to\mathbb{R}$ which is continuous on $[a,b]$ with $F'(x)=f(x)$ for all $x\in(a,b)$.

**Proof** Let $F(x)=\displaystyle\int_a^x f(t)dt$.

**Example** If $g(x)=\int_0^x\sqrt{1+t^4}\,dt$, then $g'(x)=\sqrt{1+x^4}$.

The result of the following proposition is called *integration by substitution.*

**Proposition** Suppose $I$ is an open interval, $\varphi : I \to \mathbb{R}$, $[a,b] \subset I$, and $\varphi'$ is continuous on $[a,b]$. If $f : \varphi([a,b]) \to \mathbb{R}$ is continuous, then

$$\int_{\varphi(a)}^{\varphi(b)} f(u)du = \int_a^b f(\varphi(x))\varphi'(x)dx.$$

**Proof** Suppose $\varphi(a) \leq \varphi(b)$ and let $F$ be a function which is continuous on $[\varphi(a), \varphi(b)]$ with $F'(u) = f(u)$ for every $u \in (\varphi(a), \varphi(b))$. Let $g = F \circ \varphi$. Then

$$g'(x) = F'(\varphi(x))\varphi'(x) = f(\varphi(x))\varphi'(x),$$

so

$$\int_a^b f(\varphi(x))\varphi'(x)dx = g(b) - g(a) = F(\varphi(b)) - F(\varphi(a)) = \int_{\varphi(a)}^{\varphi(b)} f(u)du.$$

If $\varphi(a) > \varphi(b)$, then

$$\begin{aligned}\int_{\varphi(a)}^{\varphi(b)} f(u)du &= -\int_{\varphi(b)}^{\varphi(a)} f(u)du \\ &= -(F(\varphi(a)) - F(\varphi(b))) \\ &= F(\varphi(b)) - F(\varphi(a)) \\ &= g(b) - g(a) \\ &= \int_a^b f(\varphi(x))\varphi'(x)dx.\end{aligned}$$

**Exercise 23.2.1**
Evaluate $\int_0^1 u\sqrt{u+1}\,du$ using (1) integration by parts and (2) substitution.

**Exercise 23.2.2**
Suppose $\varphi : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$ and periodic with period 1 (that is, $\varphi(x+1) = \varphi(x)$ for every $x \in \mathbb{R}$). Show that for any continuous function $f : \mathbb{R} \to \mathbb{R}$,

$$\int_0^1 f(\varphi(x))\varphi'(x)dx = 0.$$

**Exercise 23.2.3**
Suppose $f$ is continuous on $[a,b]$. Show that there exists a point $c \in [a,b]$ such that

$$\int_a^b f = f(c)(b-a).$$

This is the *Integral Mean Value Theorem.*

**Exercise 23.2.4**
Suppose $f$ and $g$ are continuous on $[a, b]$ and $g(x) > 0$ for all $x \in [a, b]$. Show that there exists a point $c \in [a, b]$ such that

$$\int_a^b fg = f(c) \int_a^b g.$$

This is the *Generalized Integral Mean Value Theorem.*

**Exercise 23.2.5**
Suppose $f$ is integrable on $[a, b]$ for any $b > a$. If

$$\lim_{b \to +\infty} \int_a^b f$$

exists, then we define

$$\int_a^{+\infty} f = \lim_{b \to +\infty} \int_a^b f.$$

Show that if $f(x) \geq 0$ for all $x \geq a$, $f(x) \leq g(x)$ for all $x \geq a$, and $\int_a^{+\infty} g$ exists, then $\int_a^{+\infty} f$ exists.

## 23.3 Taylor's theorem revisited

The following is another version of Taylor's theorem.

**Theorem** Suppose $f \in C^{(n+1)}(a, b)$, $\alpha \in (a, b)$, and

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(\alpha)}{k!}(x - \alpha)^k.$$

Then, for any $x \in (a, b)$,

$$f(x) = P_n(x) + \int_\alpha^x \frac{f^{(n+1)}(t)}{n!}(x - t)^n dt.$$

**Proof** By the fundamental theorem of calculus, we have

$$\int_\alpha^x f'(t)dt = f(x) - f(\alpha),$$

which implies that

$$f(x) = f(\alpha) + \int_\alpha^x f'(t)dt.$$

Hence the theorem holds for $n = 0$. Suppose the result holds for $n = k - 1$, that is,

$$f(x) = P_{k-1}(x) + \int_\alpha^x \frac{f^{(k)}(t)}{(k-1)!}(x - t)^{k-1} dt.$$

Let $F(t) = f^{(k)}(t)$, $F'(t) = f^{(k+1)}(t)$, $g(t) = \frac{(x-t)^{k-1}}{(k-1)!}$, and $G(t) = -\frac{(x-t)^k}{k!}$. Then

$$\begin{aligned}\int_\alpha^x \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1}dt &= \int_\alpha^x F(t)g(t)dt \\ &= F(x)G(x) - F(\alpha)G(\alpha) + \int_\alpha^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k dt \\ &= \frac{f^{(k)}(\alpha)(x-\alpha)^k}{k!} + \int_\alpha^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k dt.\end{aligned}$$

Hence

$$f(x) = P_k(x) + \int_\alpha^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k dt,$$

and so the theorem holds for $n = k$.

**Exercise 23.3.1**
Under the conditions of Taylor's theorem as just stated, show that

$$\int_\alpha^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt = \frac{f^{(n+1)}(\gamma)}{n!}(x-\gamma)^n(x-\alpha)$$

for some $\gamma$ between $\alpha$ and $x$. This is called the Cauchy form of the remainder.

**Exercise 23.3.2**
Under the conditions of Taylor's theorem as just stated, show that

$$\int_\alpha^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt = \frac{f^{(n+1)}(\gamma)}{(n+1)!}(x-\alpha)^{n+1}$$

for some $\gamma$ between $\alpha$ and $x$. This is called the Lagrange form of the remainder. (Note that this is the form of the remainder we derived earlier, although under slightly more general assumptions.)