Lecture 22: More Properties of Integrals

22.1 More Properties of integrals

Proposition If f is integrable on $[a, b]$ with $f(x) \geq 0$ for all $x \in [a, b]$, then

$$\int_a^b f \geq 0.$$

Proof The result follow from the fact that $L(f, P) \geq 0$ for any partition P of $[a, b]$.

Proposition Suppose f and g are both integrable on $[a, b]$. If $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f \leq \int_a^b g.$$

Proof Since $g(x) - f(x) \geq 0$ for all $x \in [a, b]$, we have

$$\int_a^b g - \int_a^b f = \int_a^b (g - f) \geq 0$$

by the previous proposition.

Proposition Suppose f is integrable on $[a, b]$, $m \in \mathbb{R}$, $M \in \mathbb{R}$, and $m \leq f(x) \leq M$ for all $x \in [a, b]$. Then

$$m(b - a) \leq \int_a^b f \leq M(b - a).$$

Proof It follows from the previous proposition that

$$m(b - a) = \int_a^b m dx \leq \int_a^b f(x) dx \leq \int_a^b M dx = M(b - a).$$

Exercise 22.1.1
Show that

$$1 \leq \int_{-1}^1 \frac{1}{1 + x^2} dx \leq 2.$$

Exercise 22.1.2
Suppose f is continuous on $[0, 1]$, differentiable on $(0, 1)$, $f(0) = 0$, and $|f'(x)| \leq 1$ for all $x \in (0, 1)$. Show that

$$-\frac{1}{2} \leq \int_0^1 f \leq \frac{1}{2}.$$

Exercise 22.1.3
Suppose f is integrable on $[a, b]$ and define $F : (a, b) \to \mathbb{R}$ by

$$F(x) = \int_a^x f.$$

Show that if $x, y \in (a, b)$, $x < y$, then there exists $\alpha \in \mathbb{R}$ such that

$$|F(y) - F(x)| \leq \alpha(y - x).$$

Proposition Suppose g is integrable on $[a, b]$, $g([a, b]) \subset [c, d]$, and $f : [c, d] \to \mathbb{R}$ is continuous. If $h = f \circ g$, then h is integrable on $[a, b]$.

Proof Let $\epsilon > 0$ be given. Let

$$K > \sup\{f(x) : x \in [c, d]\} - \inf\{f(x) : x \in [c, d]\}$$

and choose $\delta > 0$ so that $\delta < \epsilon$ and

$$|f(x) - f(y)| < \frac{\epsilon}{2(b - a)}$$

whenever $|x - y| < \delta$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a a partition of $[a, b]$ such that

$$U(g, P) - L(g, P) < \frac{\delta^2}{2K}.$$

For $i = 1, 2, \ldots, n$, let

$$m_i = \inf\{g(x) : x_{i-1} \leq x \leq x_i\},$$
$$M_i = \sup\{g(x) : x_{i-1} \leq x \leq x_i\},$$
$$w_i = \inf\{h(x) : x_{i-1} \leq x \leq x_i\},$$

and

$$W_i = \sup\{h(x) : x_{i-1} \leq x \leq x_i\}.$$

Finally, let

$$I = \{i : i \in \mathbb{Z}, 1 \leq i \leq n, M_i - m_i < \delta\}$$

and

$$J = \{i : i \in \mathbb{Z}, 1 \leq i \leq n, M_i - m_i \geq \delta\}.$$

Note that

$$\begin{aligned}\delta \sum_{i\in J}(x_i - x_{i-1}) &\le \sum_{i\in J}(M_i - m_i)(x_i - x_{i-1}) \\ &\le \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) \\ &< \frac{\delta^2}{2K},\end{aligned}$$

from which it follows that

$$\sum_{i\in J}(x_i - x_{i-1}) < \frac{\delta}{2K}.$$

Then

$$\begin{aligned}U(h,P) - L(h,P) &= \sum_{i\in I}(W_i - w_i)(x_i - x_{i-1}) + \sum_{i\in J}(W_i - w_i)(x_i - xi - 1) \\ &< \frac{\epsilon}{2(b-a)}\sum_{i\in I}(x_i - x_{i-1}) + K\sum_{i\in J}(x_i - x_{i-1}) \\ &< \frac{\epsilon}{2} + \frac{\delta}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Thus h is integrable on $[a,b]$.

Proposition Suppose f and g are both integrable on $[a,b]$. Then fg is integrable on $[a,b]$.

Proof Since f and g are both integrable, then both $f+g$ and $f-g$ are integrable. Hence, by the previous proposition, both $(f+g)^2$ and $(f-g)^2$ are integrable. Thus

$$\frac{1}{4}\Big((f+g)^2 - (f-g)^2)\Big) = fg$$

is integrable on $[a,b]$.

Proposition Suppose f is integrable on $[a,b]$. Then $|f|$ is integrable on $[a,b]$ and

$$\Big|\int_a^b f\Big| \le \int_a^b |f|.$$

Proof The integrabilty of $|f|$ follows from a previous proposition. The inequality follows from the fact that

$$-|f(x)| \le f(x) \le |f(x)|$$

for all $x \in [a,b]$. Hence

$$-\int_a^b |f| \le \int_a^b f \le \int_a^b |f|,$$

from which the result follows.

Exercise 22.1.4
Either prove the following statement or show it is false by finding a counterexample: If $f : [0, 1] \to \mathbb{R}$ is bounded and f^2 is integrable on $[0, 1]$, then f is integrable on $[0, 1]$.

22.2 Extended definitions

Definition If f is integrable on $[a, b]$, then we define

$$\int_b^a f = -\int_a^b f.$$

Moreover, if f is a function defined at a point $a \in \mathbb{R}$, we define $\int_a^a f = 0$.

Exercise 22.2.1
Suppose f is integrable on a closed interval containing the points a, b, and c. Show that

$$\int_a^b f = \int_a^c f + \int_c^b f.$$