Lecture 21: Properties of Integrals

21.1 Properties of integrals

Exercise 21.1.1
Suppose $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$. Show that

$$\sup\{f(x) + g(x) : x \in D\} \leq \sup\{f(x) : x \in D\} + \sup\{g(x) : x \in D\}$$

and

$$\inf\{f(x) + g(x) : x \in D\} \geq \inf\{f(x) : x \in D\} + \inf\{g(x) : x \in D\}$$

Find examples for which the inequalities are strict.

Proposition Suppose f and g are both integrable on $[a, b]$. Then $f + g$ is integrable on $[a, b]$ and

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

Proof Given $\epsilon > 0$, let P_1 and P_2 be partitions of $[a, b]$ with

$$U(f, P_1) - L(f, P_1) < \frac{\epsilon}{2}$$

and

$$U(g, P_2) - L(g, P_2) < \frac{\epsilon}{2}.$$

Let $P = P_1 \cup P_2$. By the previous exercise,

$$U(f + g, P) \leq U(f, P) + U(g, P)$$

and

$$L(f + g, P) \geq L(f, P) + L(g, P).$$

Hence

$$\begin{aligned} U(f + g, P) - L(f + g, P) &\leq (U(f, P) + U(g, P)) - (L(f, P) + L(g, P)) \\ &= (U(f, P) - L(f, P)) + (U(g, P) - L(g, P)) \\ &\leq (U(f, P_1) - L(f, P_1)) + (U(g, P_2) - L(g,_2 P)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence $f + g$ is integrable on $[a, b]$. Moreover,

$$\begin{aligned} \int_a^b (f + g) &\leq U(f + g, P) \\ &\leq U(f, P) + U(g, P) \\ &\leq \left(\int_a^b f + \frac{\epsilon}{2}\right) + \left(\int_a^b g + \frac{\epsilon}{2}\right) \\ &= \int_a^b f + \int_a^b g + \epsilon \end{aligned}$$

and

$$\begin{aligned}\int_a^b (f+g) &\geq L(f+g,P)\\ &\geq L(f,P)+L(g,P)\\ &\geq \left(\int_a^b f - \frac{\epsilon}{2}\right) + \left(\int_a^b g - \frac{\epsilon}{2}\right)\\ &= \int_a^b f + \int_a^b g - \epsilon.\end{aligned}$$

Since the preceding inequalities hold for any $\epsilon > 0$, it follows that

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

Exercise 21.1.2
Suppose $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are both bounded. Show that

$$\overline{\int_a^b}(f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g.$$

Find an example for which the inequality is strict.

Exercise 21.1.3
Find an example to show that $f+g$ may be integrable on $[a,b]$ even though neither f nor g is integrable on$[a,b]$.

Proposition If f is integrable on $[a,b]$ and $\alpha \in \mathbb{R}$, then αf is integrable on $[a,b]$ and

$$\int_a^b \alpha f = \alpha \int_a^b f.$$

Exercise 21.1.4
Prove the previous proposition.

Proposition Suppose $f : [a,b] \to \mathbb{R}$ is bounded and $c \in (a,b)$. Then f is integrable on $[a,b]$ if and only if f is integrable on both $[a,c]$ and $[c,b]$.

Proof Suppose f is integrable on $[a,b]$. Given $\epsilon > 0$, let Q be a partition of $[a,b]$ such that

$$U(f,Q) - L(f,Q) < \epsilon.$$

Let $P = Q \cup \{c\}$, $P_1 = P \cap [a,c]$, and $P_2 \cap [c,b]$. Then

$$\begin{aligned}(U(f,P_1)-L(f,P_1)) + (U(f,P_2)-L(f,P_2)) &= (U(f,P_1)+U(f,P_2))\\ &\quad - (L(f,P_1)+L(f,P_2))\\ &= U(f,P)-L(f,P)\\ &\leq U(f,Q)-L(f,Q)\\ &< \epsilon.\end{aligned}$$

Thus we must have both

$$U(f, P_1) - L(f, P_1) < \epsilon$$

and

$$U(f, P_2) - L(f, P_2) < \epsilon.$$

Hence f is integrable on both $[a, c]$ and $[c, b]$.

Now suppose f is integrable on both $[a, c]$ and $[c, b]$. Given $\epsilon > 0$, let P_1 and P_2 be partitions of $[a, c]$ and $[c, b]$, respectively, such that

$$U(f, P_1) - L(f, P_1) < \frac{\epsilon}{2}$$

and

$$U(f, P_2) - L(f, P_2) < \frac{\epsilon}{2}.$$

Let $P = P_1 \cup P_2$. Then P is a partition of $[a, b]$ and

$$\begin{aligned} U(f, P) - L(f, P) &= (U(f, P_1) + U(f, P_2)) - (L(f, P_1) + L(f, P_2)) \\ &= (U(f, P_1) - L(f, P_1)) + (U(f, P_2) - L(f, P_2)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus f is integrable on $[a, b]$.

Proposition Suppose f is integrable on $[a, b]$ and $c \in (a, b)$. Then

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof If P and Q are partitions of $[a, c]$ and $[c, b]$, respectively, then

$$U(f, P) + U(f, Q) = U(f, P \cup Q) \geq \int_a^b f.$$

Thus

$$U(f, P) \geq \int_a^b f - U(f, Q),$$

so

$$\int_a^c f = \overline{\int_a^c} f \geq \int_a^b f - U(f, Q).$$

Hence

$$U(f, Q) \geq \int_a^b f - \int_a^c f,$$

so

$$\int_c^b f = \overline{\int_c^b} f \geq \int_a^b f - \int_a^c f.$$

Thus

$$\int_a^c f + \int_c^b f \geq \int_a^b f.$$

Similarly, if P and Q are partitions of $[a, c]$ and $[c, b]$, respectively, then

$$L(f, P) + L(f, Q) = L(f, P \cup Q) \leq \int_a^b f.$$

Thus

$$L(f, P) \leq \int_a^b f - L(f, Q),$$

so

$$\int_a^c f = \underline{\int_a^c} f \leq \int_a^b f - L(f, Q).$$

Hence

$$L(f, Q) \leq \int_a^b f - \int_a^c f,$$

so

$$\int_c^b f = \underline{\int_c^b} f \leq \int_a^b f - \int_a^c f.$$

Thus

$$\int_a^c f + \int_c^b f \leq \int_a^b f.$$

Hence

$$\int_a^c f + \int_c^b f = \int_a^b f.$$

Exercise 21.1.5

Suppose $f : [a, b] \to \mathbb{R}$ is bounded and B is a finite subset of (a, b). Show that if f is continuous on $[a, b] \setminus B$, then f is integrable on $[a, b]$.