Lecture 20: Integrability Conditions

20.1 Integrability conditions

Proposition If $f : [a, b] \to \mathbb{R}$ is monotonic, then f is integrable on $[a, b]$.

Proof Suppose f is nondecreasing.Given $\epsilon > 0$, let $n \in Z^+$ be large enough that

$$\frac{(f(b) - f(a))(b - a)}{n} < \epsilon.$$

For $i = 0, 1, \ldots, n$, let

$$x_i = a + \frac{(b - a)i}{n}.$$

Let $P = \{x_0, x_1, \ldots, x_n\}$. Then

$$\begin{aligned}
U(f, P) - L(f, P) &= \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) - \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \frac{b - a}{n} \\
&= \frac{b - a}{n}((f(x_1) - f(x_0)) + (f(x_2) - f(x_1)) + \cdots \\
&\quad + (f(x_{n-1}) - f(x_{n-2})) + (f(x_n) - f(x_{n-1}))) \\
&= \frac{b - a}{n}(f(b) - f(a)) < \epsilon.
\end{aligned}$$

Hence f is integrable on $[a, b]$.

Example Let $\varphi : \mathbb{Q} \cap [0, 1] \to \mathbb{Z}^+$ be a one-to-one correspondence. Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \sum_{\substack{q \in \mathbb{Q} \cap [0,1] \\ q \le x}} \frac{1}{2^{\varphi(q)}}.$$

Then f is increasing on $[0, 1]$, and hence integrable on $[0, 1]$.

Proposition If $f : [a, b] \to \mathbb{R}$ is continuous, then f is integrable on $[a, b]$.

Proof Given $\epsilon > 0$, let

$$\gamma = \frac{\epsilon}{b - a}.$$

Since f is uniformly continuous on $[a, b]$, we may choose $\delta > 0$ such that

$$|f(x) - f(y)| < \gamma$$

whenever $|x - y| < \delta$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition with

$$\sup\{|x_i - x_{i-1}| : i = 1, 2, \ldots, n\} < \delta.$$

If, for $i = 1, 2, \ldots, n$,

$$m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\}$$

and

$$M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\},$$

then $M_i - m_i < \gamma$. Hence

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) - \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\ &< \gamma \sum_{i=1}^{n} (x_i - x_{i-1}) \\ &= \gamma(b-a) = \epsilon. \end{aligned}$$

Thus f is integrable on $[a, b]$.

Exercise 20.1.1
Suppose $f : [a, b] \to \mathbb{R}$ is bounded and let $c \in [a, b]$. Show that if f is continuous on $[a, b] \setminus \{c\}$, then f is integrable on $[a, b]$.

Exercise 20.1.2
Suppose f is continuous on $[a, b]$ with $f(x) \ge 0$ for all $x \in [a, b]$. Show that if $\int_a^b f = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

Exercise 20.1.3
Suppose f is continuous on $[a, b]$. For $i = 0, 1, \ldots, n$, $n \in \mathbb{Z}^+$, let

$$x_i = a + \frac{(b-a)i}{n}$$

and, for $i = 1, 2, \ldots, n$, let $c_i \in [x_{i-1}, x_i]$. Show that

$$\int_a^b f = \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=1}^{n} f(c_i).$$

(The approximation

$$\int_a^b f \approx \frac{b-a}{n} \sum_{i=1}^{n} f(c_i)$$

is called the *right-hand rule approximation* if $c_i = x_i$, the *left-hand rule approximation* if $c_i = x_{i-1}$, and the *midpoint rule approximation* if

$$c_i = \frac{x_{i-1} + x_i}{2}.$$

These are basic ingredients in creating numerical approximations to integrals.)