Lecture 19: The Integral

19.1 Upper and lower integrals

Definition Given a closed interval $[a, b] \subset \mathbb{R}$, any finite subset of $[a, b]$ which includes both a and b is called a *partition* of $[a, b]$.

For convenience, whenever we consider a partition P of an interval $[a, b]$ with $|P| = n + 1$, we will index the elements in increasing order, starting with 0. That is, if $P = \{x_0, x_1, \ldots, x_n\}$, then

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Definition Suppose $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$ and $f : [a, b] \to \mathbb{R}$ is bounded. For $i = 1, 2, \ldots, n$, let

$$m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$$

and

$$M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}.$$

We call

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

the *lower sum* of f determined by P and

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

the *upper sum* of f determined by P.

Definition If P_1 and P_2 are both partitions of $[a, b]$ and $P_1 \subset P_2$, then we call P_2 a *refinement* of P_1.

Definition If P_1 and P_2 are both partitions of $[a, b]$, then the partition $P = P_1 \cup P_2$ is called the *common refinement* of P_1 and P_2.

Lemma Suppose $f : [a, b] \to \mathbb{R}$ is bounded and $P_1 = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$. Moreover, suppose $s \in (a, b)$ with $s \notin P_1$. If $P_2 = P_1 \cup \{s\}$, then $L(f, P_1) \leq L(f, P_2)$ and $U(f, P_2) \leq U(f, P_1)$.

Proof Suppose $x_{i-1} < s < x_i$ and let

$$\begin{aligned}
w_1 &= \inf\{f(x) : x_{i-1} \leq x \leq s\}, \\
W_1 &= \sup\{f(x) : x_{i-1} \leq x \leq s\}, \\
w_2 &= \inf\{f(x) : s \leq x \leq x_i\}, \\
W_2 &= \sup\{f(x) : s \leq x \leq x_i\}, \\
m_i &= \inf\{f(x) : x_{i-1} \leq x \leq x_i\},
\end{aligned}$$

and

$$M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}.$$

Then $w_1 \geq m_i$, $w_2 \geq m_i$, $W_1 \leq M_i$, and $W_2 \leq M_i$. Hence

$$\begin{aligned} L(f,P_2) - L(f,P_1) &= w_1(s - x_{i-1}) + w_2(x_i - s) - m_i(x_i - x_{i-1}) \\ &= w_1(s - x_{i-1}) + w_2(x_i - s) - m_i(s - x_{i-1}) - m_i(x_i - s) \\ &= (w_1 - m_i)(s - x_{i-1}) + (w_2 - m_i)(x_i - s) \geq 0 \end{aligned}$$

and

$$\begin{aligned} U(f,P_1) - U(f,P_2) &= M_i(x_i - x_{i-1}) - W_1(s - x_{i-1}) - W_2(x_i - s) \\ &= M_i(s - x_{i-1}) + m_i(x_i - s) - W_1(s - x_{i-1}) - W_2(x_i - s) \\ &= (M_i - W_1)(s - x_{i-1}) + (M_i - W_2)(x_i - s) \geq 0. \end{aligned}$$

Thus $L(f,P_1) \leq L(f,P_2)$ and $U(f,P_2) \leq U(f,P_1)$.

Proposition Suppose $f : [a,b] \to \mathbb{R}$ is bounded and P_1 and P_2 are partitions of $[a,b]$. If P_2 is a refinement of P_1, then $L(f,P_1) \leq L(f,P_2)$ and $U(f,P_2) \leq U(f,P_1)$.

Proof The proposition follows immediately from repeated use of the previous lemma.

Proposition Suppose $f : [a,b] \to \mathbb{R}$ is bounded and P_1 and P_2 are partitions of $[a,b]$. Then $L(f,P_1) \leq U(f,P_2)$.

Proof The result follows immediately from the definitions if $P_1 = P_2$. Otherwise, let P be the common refinement of P_1 and P_2. Then

$$L(f,P_1) \leq L(f,P) \leq U(f,P) \leq U(f,P_2).$$

Definition Suppose $f : [a,b] \to \mathbb{R}$ is bounded. We call

$$\underline{\int_a^b} f = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}$$

the *lower integral* of f over $[a,b]$ and

$$\overline{\int_a^b} f = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}$$

the *upper integral* of f over $[a,b]$.

Note that both the lower sum and the upper sum are finite real numbers since the lower sums are all bounded above by any upper sum and the upper sums are all bounded below by any lower sum.

Proposition Suppose $f : [a,b] \to \mathbb{R}$ is bounded. Then

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

Proof Let P be a partition of $[a,b]$. Then for any partition Q of $[a,b]$, we have $L(f,Q) \leq U(f,P)$. Hence $U(f,P)$ is an upper bound for any lower sum, and so

$$\underline{\int_a^b} f \leq U(f,P).$$

But this shows that the lower integral is a lower bound for any upper sum. Hence

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

19.2 Integrals

Definition Suppose $f : [a,b] \to \mathbb{R}$ is bounded. We say f is *integrable* on $[a,b]$ if

$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

If f is integrable, we call the common value of the upper and lower integrals the *integral* of f over $[a,b]$, denoted

$$\int_a^b f.$$

That is, if f is integrable on $[a,b]$,

$$\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f.$$

Example Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

For any partition $P = \{x_0, x_1, \ldots, x_n\}$, we have

$$L(f,P) = \sum_{i=1}^{n} 0(x_i - x_{i-1}) = 0$$

and

$$U(f,P) = \sum_{i=1}^{n}(x_i - x_{i-1}) = x_n - x_0 = 1.$$

Thus

$$\underline{\int_0^1} f = 0$$

and

$$\overline{\int_0^1} f = 1.$$

Hence f is not integrable on $[0,1]$.

Example Let $\alpha \in \mathbb{R}$ and define $f : [a,b] \to \mathbb{R}$ by $f(x) = \alpha$ for all $x \in [a,b]$. For any partition $P = \{x_0, x_1, \ldots, x_n\}$, we have

$$L(f,P) = \sum_{i=1}^{n} \alpha(x_i - x_{i-1}) = \alpha(x_n - x_0) = \alpha(b-a)$$

and

$$U(f,P) = \sum_{i=1}^{n} \alpha(x_i - x_{i-1}) = \alpha(x_n - x_0) = \alpha(b-a).$$

Thus

$$\underline{\int_a^b} f = \alpha(b-a)$$

and

$$\overline{\int_a^b} f = \alpha(b-a).$$

Hence f is integrable on $[a,b]$ and

$$\int_a^b f = \alpha(b-a).$$

Theorem Suppose $f : [a,b] \to \mathbb{R}$ is bounded. Then f is integrable on $[a,b]$ if and only if for every $\epsilon > 0$ there exists a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \epsilon.$$

Proof If f is integrable on $[a,b]$ and $\epsilon > 0$, then we may choose partitions P_1 and P_2 such that

$$\int_a^b f - L(f,P_1) < \frac{\epsilon}{2}$$

and

$$U(f, P_2) - \int_a^b f < \frac{\epsilon}{2}.$$

Let P be the common refinement of P_1 and P_2. Then

$$U(f,P)-L(f,P) \leq U(f,P_2)-L(f,P_1) = (U(f,P_2)-\int_a^b f)+(\int_a^b f-L(f,P_1)) < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

Now suppose for every $\epsilon > 0$ there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon.$$

Suppose

$$\underline{\int_a^b} f < \overline{\int_a^b} f.$$

If

$$\epsilon = \overline{\int_a^b} f - \underline{\int_a^b} f,$$

then for any partition P of $[a, b]$ we have

$$U(f, P) - L(f, P) \geq \overline{\int_a^b} f - \underline{\int_a^b} f = \epsilon.$$

Since this contradicts our assumption, we must have

$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

That is, f is integrable on $[a, b]$.

Example Suppose $f : [0, 1] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0, & x \neq \frac{1}{2}, \\ 1, & x = \frac{1}{2}. \end{cases}$$

If P is a partition of $[0, 1]$, then clearly $L(f, P) = 0$. Given $\epsilon > 0$, let

$$P = \{0, \frac{1}{2} - \frac{\epsilon}{4}, \frac{1}{2} + \frac{\epsilon}{4}, 1\}.$$

Then

$$U(f, P) = (\frac{1}{2} + \frac{\epsilon}{4}) - (\frac{1}{2} - \frac{\epsilon}{4}) = \frac{\epsilon}{2} < \epsilon.$$

Hence $U(f,P) - L(f,P) < \epsilon$, so f is integrable on $[0,1]$. Moreover,

$$\int_0^1 f = 0.$$

Exercise 19.2.1
For $n \in \mathbb{Z}^+$ let $a_1, a_2, \ldots, a_n$ be points in $(0,1)$. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x = a_i \text{ for some } i, \\ 0, & \text{otherwise.} \end{cases}$$

Show that $\int_0^1 f = 0$.

Exercise 19.2.2
Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x \text{ is rational and } x = \frac{p}{q}, \\ \\ 0, & \text{if } x \text{ is irrational,} \end{cases}$$

where p and q are taken to be relatively prime integers with $q > 0$, and we take $q = 1$ when $x = 0$. Show that $\int_0^1 f = 0$.

Exercise 19.2.3
Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = x$ and, for $n \in \mathbb{Z}^+$, let $P = \{x_0, x_1, \ldots, x_n\}$ be the partition of $[0,1]$ with

$$x_i = \frac{i}{n}, i = 0, 1, \ldots, n.$$

Show that

$$U(f,P) - L(f,P) = \frac{1}{n},$$

and hence conclude that f is integrable on $[0,1]$. Show that

$$\int_0^1 f = \frac{1}{2}.$$

Exercise 19.2.4
Define $f : [1,2] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Show that f is not integrable on $[1,2]$.

Exercise 19.2.5
Suppose f is integrable on $[a, b]$ and $f(x) \geq m$ for all $x \in [a, b]$ and $f(x) \leq M$ for all $x \in [a, b]$. Show that

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

19.3 Notation and terminology

The definition of the integral given in this section is due to Darboux. It may be shown to be equivalent to the integral defined by Riemann. Hence functions that are integrable in the sense of this lecture are referred to as *Riemann integrable* functions and the integral is called the *Riemann integral.* This is in distinction to the *Lebesgue integral*, part of a more general theory of integration.

The integral of this lecture is also called the *definite integral*, as opposed to an *indefinite integral*, the latter being a name given to an *antiderivative* (a function whose derivative is equal to a given function).

If f is integrable on $[a, b]$, then we will also denote

$$\int_a^b f$$

by

$$\int_a^b f(x)dx.$$

The variable x in the latter is a "dummy" variable; we might just as well write

$$\int_a^b f(t)dt$$

or

$$\int_a^b f(s)ds.$$

For example, if $f : [0, 1] \to \mathbb{R}$ is defined by $f(x) = x^2$, then

$$\int_0^1 f = \int_0^1 x^2 dx = \int_0^1 t^2 dt.$$