Lecture 18: Taylor's Theorem

18.1 Derivatives of higher order

Definition Suppose f is differentiable on an open interval I and f' is differentiable at $a \in I$. We call the derivative of f' at a the *second derivative* of f at a, which we denote $f''(a)$.

By continued differentiation, we may define the *higher order derivatives* f''', f'''', and so on. In general, for any integer n, $n \geq 0$, we let $f^{(n)}$ denote the nth derivative of f, where $f^{(0)}$ denotes f.

Exercise 18.1.1
Suppose $D \subset \mathbb{R}$, a is an interior point of D, $f : D \to \mathbb{R}$, and $f''(a)$ exists. Show that

$$\lim_{h\to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2} = f''(a).$$

Find an example to illustrate that this limit may exist even if $f''(a)$ does not exist.

For any open interval (a, b), where a and b are extended real numbers, we let $C^{(n)}(a, b)$ denote the set of all functions f with the property that each of $f, f^{(1)}, f^{(2)}, \ldots, f^{(n)}$ is defined and continuous on (a, b).

18.2 Taylor's theorem

The following result is known as Taylor's theorem.

Theorem Suppose $f \in C^{(n)}(a, b)$ and $f^{(n)}$ is differentiable on (a, b). Let $\alpha, \beta \in (a, b)$ with $\alpha \neq \beta$, and let

$$P(x) = f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\alpha)}{2}(x-\alpha)^2 + \cdots + \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n = \sum_{k=0}^{n} \frac{f^{(k)}(\alpha)}{k!}(x-\alpha)^k.$$

Then there exists a point γ between α and β such that

$$f(\beta) = P(\beta) + \frac{f^{(n+1)}(\gamma)}{(n+1)!}(\beta - \alpha)^{n+1}.$$

Proof First note that $P^{(k)}(\alpha) = f^{(k)}(\alpha)$ for $k = 0, 1, \ldots, n$. Let

$$M = \frac{f(\beta) - P(\beta)}{(\beta - \alpha)^{n+1}}.$$

Then

$$f(\beta) = P(\beta) + M(\beta - \alpha)^{n+1}.$$

We need to show that

$$M = \frac{f^{(n+1)}(\gamma)}{(n+1)!}$$

for some γ between α and β. Let

$$g(x) = f(x) - P(x) - M(x-\alpha)^{n+1}.$$

Then, for $k = 0, 1, \ldots, n$,

$$g^{(k)}(\alpha) = f^{(k)}(\alpha) - P^{(k)}(\alpha) = 0.$$

Now $g(\beta) = 0$, so, by Rolle's theorem, there exists γ_1 between α and β such that $g'(\gamma_1) = 0$. Using Rolle's theorem again, we see that there exists γ_2 between α and γ_1 such that $g''(\gamma_2) = 0$. Continuing for $n+1$ steps, we find γ_{n+1} between α and γ_n (and hence between α and β) such that $g^{(n+1)}(\gamma_{n+1}) = 0$. Hence

$$0 = g^{(n+1)}(\gamma_{n+1}) = f^{(n+1)}(\gamma_{n+1}) - (n+1)!M.$$

Letting $\gamma = \gamma_{n+1}$, we have

$$M = \frac{f^{(n+1)}(\gamma)}{(n+1)!},$$

as required.

The polynomial P in the statement of Taylor's theorem is called the *Taylor polynomial* of order n for f at α.

Example Let $f(x) = \sqrt{x}$. Then the 4th order Taylor polynomial for f at 1 is

$$P(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4.$$

By Taylor's theorem, for any $x > 0$ there exists γ between 1 and x such that

$$\sqrt{x} = P(x) + \frac{105}{(32)(5!)\gamma^{\frac{9}{2}}}(x-1)^5 = P(x) + \frac{7}{256\gamma^{\frac{9}{2}}}(x-1)^5.$$

For example,

$$\sqrt{1.2} = P(1.2) + \frac{7}{256\gamma^{\frac{9}{2}}}(1.2-1)^5 = P(1.2) + \frac{7}{256\gamma^{\frac{9}{2}}}(0.2)^5 = P(1.2) + \frac{7}{800000\gamma^{\frac{9}{2}}},$$

for some γ with $1 < \gamma < 1.2$. Hence $P(1.2)$ underestimates $\sqrt{1.2}$ by a value which is no larger than $\frac{7}{800000}$. Note that

$$P(1.2) = \frac{17527}{16000} = 1.0954375$$

and

$$\frac{7}{800000} = 0.00000875.$$

So $\sqrt{1.2}$ lies between 1.0954375 and 1.09544625.

Exercise 18.2.1
Use the 5th order Taylor polynomial for $f(x) = \sqrt{x}$ at 1 to estimate $\sqrt{1.2}$. Is this an underestimate or an overestimate? Find an upper bound for the largest amount by which the estimate and $\sqrt{1.2}$ differ.

Exercise 18.2.2
Find the 3rd order Taylor polynomial for $f(x) = \sqrt[3]{1+x}$ at 0 and use it to estimate $\sqrt[3]{1.1}$. Is this an underestimate or an overestimate? Find an upper bound for the largest amount by which the estimate and $\sqrt[3]{1.1}$ differ.

Exercise 18.2.3
Suppose $f \in C^{(2)}(a,b)$. Use Taylor's theorem to show that

$$\lim_{h \to 0} \frac{f(c+h) + f(c-h) - 2f(c)}{h^2} = f''(c)$$

for any $c \in (a,b)$.

Exercise 18.2.4
Suppose $f \in C^{(1)}(a,b)$, $c \in (a,b)$, $f'(c) = 0$, and f'' exists on (a,b) and is continuous at c. Show that f has a local maximum at c if $f''(c) < 0$ and a local minimum at c if $f''(c) > 0$.