Lecture 17: Mean Value Theorem

17.1 Rolle's theorem

Definition We say f is *differentiable* on an open interval I if f is differentiable at every point $a \in I$.

Definition Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. We say f has a *local maximum* at a point $a \in D$ if there exists $\delta > 0$ such that $f(a) \geq f(x)$ for all $x \in (a - \delta, a + \delta) \cap D$. We say f has a *local minimum* at a point $a \in D$ if there exists $\delta > 0$ such that $f(a) \leq f(x)$ for all $x \in (a - \delta, a + \delta) \cap D$.

Proposition Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and a is an interior point of D at which f has either a local maximum or a local minimum. If f is differentiable at a, then $f'(a) = 0$.

Proof Suppose f has a local maximum at a. Choose $\delta > 0$ so that $(a - \delta, a + \delta) \subset D$ and $f(a) \geq f(x)$ for all $x \in (a - \delta, a + \delta)$. Then

$$\frac{f(x) - f(a)}{x - a} \geq 0$$

for all $x \in (a - \delta, a)$ and

$$\frac{f(x) - f(a)}{x - a} \leq 0$$

for all $x \in (a, a + \delta)$. Hence

$$\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} \geq 0$$

and

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \leq 0.$$

Hence

$$0 \leq \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = f'(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \leq 0,$$

so we must have $f'(a) = 0$.

The following theorem is known as *Rolle's theorem.*

Theorem Let $a, b \in \mathbb{R}$ and suppose f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a point $c \in (a, b)$ at which $f'(c) = 0$.

Proof By the Extreme Value Theorem, we know f attains a maximum and a minimum value on $[a, b]$. Let m be the minimum value and M the maximum value of f on $[a, b]$. If $m = M = f(a) = f(b)$, then $f(x) = m$ for all $x \in [a, b]$, and so $f'(x) = 0$ for all $x \in (a, b)$. Otherwise, one of m or M occurs at a point c in (a, b). Hence f has either a local maximum or a local minimum at c, and so $f'(c) = 0$.

Exercise 17.1.1
Suppose f is differentiable on (a, b) and $f'(x) \neq 0$ for all $x \in (a, b)$. Show that for any $x, y \in (a, b)$, $f(x) \neq f(y)$.

Exercise 17.1.2
Explain why the equation $x^5 + 10x = 5$ has exactly one solution.

Exercise 17.1.3
Let $f(x)$ be a third degree polynomial. Show that the equation $f(x) = 0$ as at least one, but no more than three, solutions.

17.2 The Mean Value Theorem

The following theorem is known as the *generalized mean value theorem.*

Theorem Let $a, b \in \mathbb{R}$. If f and g are continuous on $[a, b]$ and differentiable on (a, b), then there exists a point $c \in (a, b)$ at which

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

Proof Let

$$h(t) = (f(b) - f(a))g(t) - (g(b) - g(a))f(t).$$

Then h is continuous on $[a, b]$, differentiable on (a, b), and

$$h(a) = f(b)g(a) - f(a)g(a) - f(a)g(b) + f(a)g(a) = f(b)g(a) - f(a)g(b)$$

and

$$h(b) = f(b)g(b) - f(a)g(b) - f(b)g(b) + f(b)g(a) = f(b)g(a) - f(a)g(b).$$

Hence, by Rolle's theorem, there exists a point $c \in (a, b)$ at which $h'(c) = 0$. But then

$$0 = h'(c) = (f(b) - f(a))g'(c) - (g(b) - g(a))f'(c),$$

which implies that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

The following theorem is known as the *Mean Value Theorem.*

Theorem Let $a, b \in \mathbb{R}$. If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point $c \in (a, b)$ at which

$$f(b) - f(a) = (b - a)f'(c).$$

Proof Apply the previous result with $g(x) = x$.

Exercise 17.2.1
Prove the Mean Value Theorem using Rolle's theorem and the function

$$k(t) = f(t) - \left(\left(\frac{f(b) - f(a)}{b - a}\right)(t - a) + f(a)\right).$$

Give a geometric interpretation to k and compare it with the function h used in the proof of the generalized mean value theorem.

Exercise 17.2.2
Let $a, b \in \mathbb{R}$. Suppose f is continuous on $[a, b]$, differentiable on (a, b), and $|f'(x)| \leq M$ for all $x \in (a, b)$. Show that

$$|f(b) - f(a)| \leq M|b - a|.$$

Exercise 17.2.3
Show that for all $x > 0$, $\sqrt{1+x} < 1 + \frac{x}{2}$.

Exercise 17.2.4
Suppose I is an open interval, $f : I \to \mathbb{R}$, and $f'(x) = 0$ for all $x \in I$. Show that there exists $\alpha \in \mathbb{R}$ such that $f(x) = \alpha$ for all $x \in I$.

Exercise 17.2.5
Suppose I is an open interval, $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$, and $f'(x) = g'(x)$ for all $x \in I$. Show that there exists $\alpha \in \mathbb{R}$ such that $g(x) = f(x) + \alpha$ for all $x \in I$.

Exercise 17.2.6
Let $D = \mathbb{R} \setminus \{0\}$. Define $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ by $f(x) = x^2$ and

$$g(x) = \begin{cases} x^2, & \text{if } x < 0, \\ x^2 + 1, & \text{if } x > 0. \end{cases}$$

Show that $f'(x) = g'(x)$ for all $x \in D$, but there does not exist $\alpha \in \mathbb{R}$ such that $g(x) = f(x) + \alpha$ for all $x \in D$. Why does this not contradict the conclusion of the previous exercise?

Proposition If f is differentiable on (a, b) and $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on (a, b).

Proof Let $x, y \in (a, b)$ with $x < y$. By the Mean Value Theorem, there exists a point $c \in (x, y)$ such that

$$f(y) - f(x) = (y - x)f'(c).$$

Since $y - x > 0$ and $f'(c) > 0$, we have $f(y) > f(x)$, and so f is increasing on (a, b).

Proposition If f is differentiable on (a, b) and $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on (a, b).

Exercise 17.2.7
State and prove similar conditions for nonincreasing and nondecreasing functions.

17.3 Discontinuities of derivatives

The following theorem is sometimes called the *intermediate value theorem for derivatives.*

Theorem Suppose f is differentiable on an open interval I and $a, b \in I$. If $\lambda \in \mathbb{R}$ and either $f'(a) < \lambda < f'(b)$ or $f'(a) > \lambda > f'(b)$, then there exists $c \in (a, b)$ such that $f'(c) = \lambda$.

Proof Suppose $f'(a) < \lambda < f'(b)$ and define $g : I \to \mathbb{R}$ by $g(x) = f(x) - \lambda x$. Then g is differentiable on I, and so continuous on $[a, b]$. Let c be a point in $[a, b]$ at which g attains its minimum value. Now

$$g'(a) = f'(a) - \lambda < 0,$$

so there exists $a < t < b$ such that

$$g(t) - g(a) < 0.$$

Thus $c \neq a$. Similarly,

$$g'(b) = f'(b) - \lambda > 0,$$

so there exists $a < s < b$ such that

$$g(s) - g(b) < 0.$$

Thus $c \neq b$. Hence $c \in (a, b)$, and so $g'(c) = 0$. Thus $0 = f'(c) - \lambda$, and so $f'(c) = \lambda$.

Exercise 17.3.1
Define $g : (-1, 1) \to \mathbb{R}$ by

$$g(x) = \begin{cases} -1, & \text{if } -1 < x < 0, \\ 1, & \text{if } 0 \leq x < 1. \end{cases}$$

Does there exist a function $f : (-1, 1) \to \mathbb{R}$ such that $f'(x) = g(x)$ for all $x \in (-1, 1)$.

Exercise 17.3.2
Suppose f is differentiable on an open interval I. Show that f' cannot have any simple discontinuities in I.

Example Define $\varphi : [0, 1] \to \mathbb{R}$ by $\varphi(x) = x(2x - 1)(x - 1)$. Define $\psi : \mathbb{R} \to \mathbb{R}$ by $\psi(x) = 6x^2 - 6x + 1$. Then

$$\varphi(x) = 2x^3 - 3x^2 + x,$$

so $\varphi'(x) = \psi(x)$ for all $x \in (0, 1)$. Next define $s : \mathbb{R} \to \mathbb{R}$ by $s(x) = \varphi(x - \lfloor x \rfloor)$. Then for any $n \in \mathbb{Z}$ and $n < x < n + 1$,

$$s'(x) = \psi(x - n) = \psi(x - \lfloor x \rfloor).$$

Moreover, if x is an integer,

$$\begin{aligned} \lim_{h \to 0^+} \frac{s(x + h) - s(x)}{h} &= \lim_{h \to 0^+} \frac{\varphi(h)}{h} \\ &= \lim_{h \to 0^+} \frac{h(2h - 1)(h - 1)}{h} \\ &= \lim_{h \to 0^+} (2h - 1)(h - 1) = 1 \end{aligned}$$

and

$$\lim_{h\to 0^-}\frac{s(x+h)-s(x)}{h} = \lim_{h\to 0^-}\frac{\varphi(h+1)}{h}$$
$$= \lim_{h\to 0^-}\frac{(h+1)(2h+1)h}{h}$$
$$= \lim_{h\to 0^-}(h+1)(2h+1) = 1.$$

Thus $s'(x) = 1 = \psi(x - \lfloor x \rfloor)$ when x is an integer, and so $s'(x) = \psi(x - \lfloor x \rfloor)$ for all $x \in \mathbb{R}$.

Now $\psi(x) = 0$ if and only if $x = \frac{3-\sqrt{3}}{6}$ or $x = \frac{3+\sqrt{3}}{6}$. Since $\varphi(0) = 0$, $\varphi(\frac{3-\sqrt{3}}{6}) = \frac{1}{6\sqrt{3}}$, $\varphi(\frac{3+\sqrt{3}}{6}) = -\frac{1}{6\sqrt{3}}$, and $\varphi(1) = 0$, we see that φ attains a maximum value of $\frac{1}{6\sqrt{3}}$ and a minimum value of $-\frac{1}{6\sqrt{3}}$. Hence for any $n \in \mathbb{Z}$, $s((n, n+1)) = [-\frac{1}{6\sqrt{3}}, \frac{1}{6\sqrt{3}}]$.

Also, $\psi'(x) = 12x - 6$, so $\psi'(x) = 0$ if and only if $x = \frac{1}{2}$. Since $\psi(0) = 1$, $\psi(\frac{1}{2}) = -\frac{1}{2}$, and $\psi(1) = 1$, we see that ψ attains a maximum value of 1 and a minimum value of $-\frac{1}{2}$ on the interval $[0, 1]$. Hence for any $n \in \mathbb{Z}$, $s'((n, n+1)) = [-\frac{1}{2}, 1]$.

It follows from the preceding that neither the function $k(x) = s(\frac{1}{x})$ nor the function $g(x) = s'(\frac{1}{x})$ has a limit as x approaches 0.

Finally, let $D = \mathbb{R} \setminus 0$ and define $f : D \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 s(\frac{1}{x}), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

For $x \neq 0$, we have

$$f'(x) = x^2 s'\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + 2xs\left(\frac{1}{x}\right) = -s'\left(\frac{1}{x}\right) + 2xs\left(\frac{1}{x}\right).$$

At 0, we have

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$
$$= \lim_{h\to 0}\frac{h^2 s\left(\frac{1}{h}\right)}{h}$$
$$= \lim_{h\to 0} hs\left(\frac{1}{h}\right) = 0,$$

where the final limit follows from the squeeze theorem and the fact that s is bounded. Hence we see that f is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R}$, but f' is not continuous since $f'(x)$ does not have a limit as x approaches 0.

Exercise 17.3.3
Let D and s be as above and define $g : D \to \mathbb{R}$ by

$$g(x) = \begin{cases} x^4 s(\frac{1}{x}), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Show that g is differentiable on $\mathbb{R}$ and that g' is continuous on $\mathbb{R}$.

17.4 l'Hôpital's rule

The following result is one case of *l'Hôpital's rule.*

Theorem Suppose $a, b \in \mathbb{R}$, f and g are differentiable on (a, b), $g'(x) \neq 0$ for all $x \in (a, b)$, and

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = \lambda.$$

If $\lim_{x \to a^+} f(x) = 0$ and $\lim_{x \to a^+} g(x) = 0$, then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lambda.$$

Proof Given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\lambda - \frac{\epsilon}{2} < \frac{f'(x)}{g'(x)} < \lambda + \frac{\epsilon}{2}$$

whenever $x \in (a, a + \delta)$. Now, by the generalized mean value theorem, for any x and y with $a < x < y < a + \delta$, there exist a point $c \in (x, y)$ such that

$$\frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f'(c)}{g'(c)}.$$

Hence

$$\lambda - \frac{\epsilon}{2} < \frac{f(y) - f(x)}{g(y) - g(x)} < \lambda + \frac{\epsilon}{2}.$$

Now

$$\lim_{x \to a} \frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f(y)}{g(y)},$$

and so we have

$$\lambda - \epsilon < \lambda - \frac{\epsilon}{2} \leq \frac{f(y)}{g(y)} \leq \lambda + \frac{\epsilon}{2} < \lambda + \epsilon$$

for any $y \in (a, a + \delta)$. Hence

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lambda.$$

Exercise 17.4.1
Use l'Hôpital's rule to compute

$$\lim_{x \to 0^+} \frac{\sqrt{1 + x} - 1}{x}.$$

Exercise 17.4.2
Suppose $a, b \in \mathbb{R}$, f and g are differentiable on (a, b), $g'(x) \neq 0$ for all $x \in (a, b)$, and

$$\lim_{x \to b^-} \frac{f'(x)}{g'(x)} = \lambda.$$

Show that if $\lim_{x \to b^-} f(x) = 0$ and $\lim_{x \to b^-} g(x) = 0$, then

$$\lim_{x \to b^-} \frac{f(x)}{g(x)} = \lambda.$$