# Lecture 16: Derivatives

## 16.1 Best linear approximations and the derivative

**Definition** A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *linear* if for every $x, y \in \mathbb{R}$,

$$f(x + y) = f(x) + f(y)$$

and for every $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}$,

$$f(\alpha x) = \alpha f(x).$$

**Exercise 16.1.1**
Show that if $f : \mathbb{R} \to \mathbb{R}$ is linear, then there exists $m \in \mathbb{R}$ such that $f(x) = mx$ for all $x \in \mathbb{R}$.

**Definition** Suppose $D \in \mathbb{R}$, $f : D \to \mathbb{R}$, and $a$ is an interior point of $D$. We say $f$ is *differentiable* at $a$ if there exists a linear function $df_a : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{x \to a} \frac{f(x) - f(a) - df_a(x - a)}{x - a} = 0.$$

The function $df_a$ is called the *best linear approximation* to $f$ at $a$, or the *differential* of $f$ at $a$.

**Proposition** Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $a$ is an interior point of $D$. Then $f$ is differentiable at $a$ if and only if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists, in which case $df_a(x) = mx$ where

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

**Proof** Let $m \in \mathbb{R}$ and let $L : \mathbb{R} \to \mathbb{R}$ be the linear function $L(x) = mx$. Then

$$\frac{f(x) - f(a) - L(x - a)}{x - a} = \frac{f(x) - f(a) - m(x - a)}{x - a} = \frac{f(x) - f(a)}{x - a} - m.$$

Hence

$$\lim_{x \to a} \frac{f(x) - f(a) - L(x - a)}{x - a} = 0$$

if and only if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = m.$$

**Definition** Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, $a$ is an interior point of $D$, and $f$ is differentiable at $a$. We call

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

the *derivative* of $f$ at $a$, which we denote $f'(a)$.

Note that if $f$ is differentiable at $a$, then

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

**Definition** Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $E$ is the set of interior points of $D$ at which $f$ is differentiable. The function $f' : E \to \mathbb{R}$ defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is called the *derivative* of $f$.

**Example** Let $n \in \mathbb{Z}^+$ and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^n$. Then

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h \to 0} \frac{x^n + nx^{n-1}h + \sum_{k=2}^{n} \binom{n}{k} x^{n-k} h^k - x^n}{h} \\ &= \lim_{h \to 0} (nx^{n-1} + \sum_{k=2}^{n} \binom{n}{k} x^{n-k} h^{k-1}) \\ &= nx^{n-1}. \end{aligned}$$

**Example** Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = |x|$. Then

$$\frac{f(0+h) - f(0)}{h} = \frac{|h|}{h} = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{if } x < 0. \end{cases}$$

Hence

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = -1$$

and

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = 1.$$

Thus $f$ is not differentiable at 0.

**Exercise 16.1.2**
Show that if $c \in \mathbb{R}$ and $f(x) = c$ for all $x \in \mathbb{R}$, then $f'(x) = 0$ for all $x \in \mathbb{R}$.

**Exercise 16.1.3**
Define $f : [0, +\infty) \to [0, +\infty)$ by $f(x) = \sqrt{x}$. Show that $f' : (0, +\infty) \to (0, +\infty)$ is given by

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

**Exercise 16.1.4**
Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{if } x < 0, \\ x^2, & \text{if } x \geq 0. \end{cases}$$

Is $f$ differentiable at 0?

**Exercise 16.1.5**
Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2, & \text{if } x < 0, \\ x^3, & \text{if } x \geq 0. \end{cases}$$

Is $f$ differentiable at 0?

**Proposition** If $f$ is differentiable at $a$, then $f$ is continuous at $a$.

**Proof** If $f$ is differentiable at $a$, then

$$\lim_{x \to a}(f(x) - f(a)) = \lim_{x \to a}\left(\frac{f(x) - f(a)}{x - a}\right)(x - a) = f'(a)(0) = 0.$$

Hence $\lim_{x \to a} f(x) = f(a)$, and so $f$ is continuous at $a$.

## 16.2 The rules

**Proposition** Suppose $f$ is differentiable at $a$ and $\alpha \in \mathbb{R}$. Then $\alpha f$ is differentiable at $a$ and $(\alpha f)'(a) = \alpha f'(a)$.

**Exercise 16.2.1**
Prove the previous proposition.

**Proposition** Suppose $f$ and $g$ are both differentiable at $a$. Then $f + g$ is differentiable at $a$ and $(f + g)'(a) = f'(a) + g'(a)$.

**Exercise 16.2.2**
Prove the previous proposition.

The following proposition is called the *product rule.*

**Proposition** Suppose $f$ and $g$ are both differentiable at $a$. Then $fg$ is differentiable at $a$ and $(fg)'(a) = f(a)g'(a) + g(a)f'(a)$.

**Proof** We have

$$\begin{aligned}(fg)'(a) &= \lim_{h\to 0}\frac{f(a+h)g(a+h)-f(a)g(a)}{h}\\ &= \lim_{h\to 0}\frac{f(a+h)g(a+h)-f(a)g(a+h)+f(a)g(a+h)-f(a)g(a)}{h}\\ &= \lim_{h\to 0}\left(g(a+h)\frac{f(a+h)-f(a)}{h}+f(a)\frac{g(a+h)-g(a)}{h}\right)\\ &= g(a)f'(a)+f(a)g'(a),\end{aligned}$$

where we know $\lim_{h\to 0} g(a+h) = g(a)$ by the continuity of $g$ at $a$, which in turn follows from the assumption that $g$ is differentiable at $a$.

**Exercise 16.2.3**
Given $n \in \mathbb{Z}+$ and $f(x) = x^n$, use induction and the product rule to show that $f'(x) = nx^{n-1}$.

The following proposition is called the *quotient rule*.

**Proposition** Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$, $a$ is in the interior of $D$, and $g(x) \neq 0$ for all $x \in D$. If $f$ and $g$ are both differentiable at $a$, then $\frac{f}{g}$ is differentiable at $a$ and

$$\left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a)-f(a)g'(a)}{(g(a))^2}.$$

**Proof** We have

$$\begin{aligned}\left(\frac{f}{g}\right)'(a) &= \lim_{h\to 0}\frac{\frac{f(a+h)}{g(a+h)}-\frac{f(a)}{g(a)}}{h}\\ &= \lim_{h\to 0}\frac{f(a+h)g(a)-f(a)g(a+h)}{hg(a+h)g(a)}\\ &= \lim_{h\to 0}\frac{f(a+h)g(a)-f(a)g(a)+f(a)g(a)-f(a)g(a+h)}{hg(a+h)g(a)}\\ &= \lim_{h\to 0}\frac{g(a)\frac{f(a+h)-f(a)}{h}-f(a)\frac{g(a+h)-g(a)}{h}}{g(a+h)g(a)}\\ &= \frac{g(a)f'(a)-f(a)g'(a)}{(g(a))^2},\end{aligned}$$

where we know $\lim_{h\to 0} g(a+h) = g(a)$ by the continuity of $g$ at $a$, which in turn follows from the assumption that $g$ is differentiable at $a$.

**Exercise 16.2.4**
Show that for any integer $n \neq 0$, if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

The following proposition is called the *chain rule*.

**Proposition** Suppose $D \subset \mathbb{R}$, $E \subset \mathbb{R}$, $g : D \to \mathbb{R}$, $f : E \to \mathbb{R}$, $g(D) \subset E$, $g$ is differentiable at $a$, and $f$ is differentiable at $g(a)$. Then $f \circ g$ is differentiable at $a$ and

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

**Proof** Choose $\delta > 0$ such that $(a-\delta, a+\delta) \subset D$ and $\epsilon > 0$ such that $(g(a)-\epsilon, g(a)+\epsilon) \subset E$. Define $\varphi : (-\delta, \delta) \to \mathbb{R}$ by

$$\varphi(h) = \begin{cases} \frac{g(a+h)-g(a)-g'(a)h}{h}, & \text{if } h \neq 0, \\ 0, & \text{if } h = 0, \end{cases}$$

and $\psi : (-\epsilon, \epsilon) \to \mathbb{R}$ by

$$\psi(h) = \begin{cases} \frac{f(g(a)+h)-f(g(a))-f'(g(a))h}{h}, & \text{if } h \neq 0, \\ 0, & \text{if } h = 0. \end{cases}$$

The assumption that $g$ is differentiable at $a$ implies that $\varphi$ is continuous at 0 and the assumption that $f$ is differentiable at $g(a)$ implies that $\psi$ is continuous at 0. Moreover, note that

$$g(a+h) = h\varphi(h) + g'(a)h + g(a)$$

for $h \in (-\delta, \delta)$ and

$$f(g(a)+h)) = h\psi(h) + f'(g(a))h + f(g(a))$$

for $h \in (-\epsilon, \epsilon)$. Hence

$$f(g(a+h)) = f(h\varphi(h) + g'(a)h + g(a))$$

for $h \in (-\delta, \delta)$. Now

$$\lim_{h \to 0}(h\varphi(h) + g'(a)h) = 0,$$

so we may choose $\gamma > 0$ so that $\gamma \leq \delta$ and

$$h\varphi(h) + g'(a)h < \epsilon$$

whenever $h \in (-\gamma, \gamma)$. Then

$$f(g(a+h)) = (h\varphi(h) + g'(a)h)\psi(h\varphi(h) + g'(a)h) + f'(g(a))(h\varphi(h) + g'(a)h) + f(g(a)),$$

so

$$\begin{aligned} f(g(a+h)) - f(g(a)) &= (h\varphi(h) + g'(a)h)\psi(h\varphi(h) + g'(a)h) + f'(g(a))(h\varphi(h) + g'(a)h) \\ &= h\varphi(h)\psi(h\varphi(h) + g'(a)h) + hg'(a)\psi(h\varphi(h) + g'(a)h) \\ &\quad + f'(g(a))\varphi(h)h + f'(g(a))g'(a)h. \end{aligned}$$

Hence

$$\frac{f(g(a+h))-f(g(a))}{h} = f'(g(a))g'(a) + \varphi(h)\psi(h\varphi(h)+g'(a)h) + g'(a)\psi(h\varphi(h)+g'(a)h) + f'(g(a))\varphi(h).$$

Now

$$\lim_{h\to 0}\varphi(h) = 0,$$

$$\lim_{h\to 0}(h\varphi(h)+g'(a)h) = 0,$$

and, since $\varphi$ and $\psi$ are continuous at 0,

$$\lim_{h\to 0}\psi(h\varphi(h)+g'(a)h) = 0.$$

Thus

$$\lim_{h\to 0}\frac{f(g(a+h))-f(g(a))}{h} = f'(g(a))g'(a).$$

**Proposition** Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$ is one-to-one, $a$ is in the interior of $D$, $f^{-1}$ is continuous at $f(a)$, and $f$ is differentiable at $a$ with $f'(a) \neq 0$. Then $f^{-1}$ is differentiable at $f(a)$ and

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}.$$

**Proof** Choose $\delta > 0$ so that $(f(a)-\delta, f(a)+\delta) \subset f(D)$. For $h \in (-\delta, \delta)$, let

$$k = f^{-1}(f(a)+h) - a.$$

Then

$$f^{-1}(f(a)+h) = a+k,$$

so

$$f(a)+h = f(a+k)$$

and

$$h = f(a+k) - f(a).$$

Hence

$$\frac{f^{-1}(f(a)+k)-f^{-1}(f(a))}{h} = \frac{a+k-a}{f(a+k)-f(a)} = \frac{1}{\frac{f(a+k)-f(a)}{k}}.$$

Now if $h \to 0$, then $k \to 0$ (since $f^{-1}$ is continuous at $f(a)$), and so

$$\lim_{h\to 0}\frac{f^{-1}(f(a)+k)-f^{-1}(f(a))}{h} = \lim_{k\to 0}\frac{1}{\frac{f(a+k)-f(a)}{k}} = \frac{1}{f'(a)}.$$

**Example** For $n \in Z^+$, define $f : [0,+\infty) \to \mathbb{R}$ by $f(x) = \sqrt[n]{x}$. Then $f$ is the inverse of $g : [0,+\infty) \to \mathbb{R}$ defined by $g(x) = x^n$. Thus, for any $x \in (0,+\infty)$,

$$f'(x) = \frac{1}{g'(f(x))} = \frac{1}{n(\sqrt[n]{x})^{n-1}} = \frac{1}{n}x^{\frac{1}{n}-1}.$$

**Exercise 16.2.5**
Show that for any rational $n \neq 0$, if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.