Lecture 15: Consequences of Continuity

15.1 Intermediate Value Theorem

The following result is known as the *Intermediate Value Theorem.*

Theorem Suppose $a, b \in \mathbb{R}$, $a < b$, and $f : [a, b] \to \mathbb{R}$. If f is continuous and $s \in \mathbb{R}$ is such that either $f(a) \leq s \leq f(b)$ or $f(b) \leq s \leq f(a)$, then there exists $c \in [a, b]$ such that $f(c) = s$.

Proof Suppose $f(a) < f(b)$ and $f(a) < s < f(b)$. Let

$$c = \sup\{x : x \in [a, b], f(x) \leq s\}.$$

Suppose $f(c) < s$. Then $c < b$ and, since f is continuous at c, there exists a $\delta > 0$ such that $f(x) < s$ for all $x \in (c, c + \delta)$. But then

$$f\left(c + \frac{\delta}{2}\right) < s,$$

contradicting the definition of c. Similarly, if $f(c) > s$, then $c > a$ and there exists $\delta > 0$ such that $f(x) > s$ for all $x \in (c - \delta, c)$, again contradicting the definition of c. Hence we must have $f(c) = s$.

Example Suppose $a \in \mathbb{R}$, $a > 0$, and consider $f(x) = x^n - a$ where $n \in \mathbb{Z}$, $n > 1$. Then $f(0) = -a < 0$ and

$$\begin{aligned} f(1 + a) &= (1 + a)^n - a \\ &= 1 + na + \sum_{i=2}^{n} \binom{n}{i} a^i - a \\ &= 1 + (n - 1)a + \sum_{i=2}^{n} \binom{n}{i} a^i > 0, \end{aligned}$$

where

$$\binom{n}{i} = \frac{n!}{i!(n - i)!}.$$

Hence, by the Intermediate Value Theorem, there exists a real number $\gamma > 0$ such that $\gamma^n = a$. We call γ the *nth root* of a, and write

$$\gamma = \sqrt[n]{a},$$

or

$$\gamma = a^{\frac{1}{n}}.$$

Moreover, if $a \in \mathbb{R}$, $a < 0$, $n \in Z^+$ is odd, and γ is the nth root of $-a$, then

$$(-\gamma)^n = (-1)^n(\gamma)^n = (-1)(-a) = a.$$

That is, $-\gamma$ is the nth root of a.

Definition If $n = \frac{p}{q} \in \mathbb{Q}$ with $q \in \mathbb{Z}^+$, then we define

$$x^n = (\sqrt[q]{x})^p$$

for all real $x \geq 0$.

Exercise 15.1.1
Explain why the equation $x^5 + 4x^2 - 16 = 0$ has a solution in the interval $(0, 2)$.

Exercise 15.1.2
Give an example of a closed interval $[a, b]$, $a, b \in \mathbb{R}$ and a function $f : [a, b] \to \mathbb{R}$ which do not satisfy the conclusion of the Intermediate Value Theorem.

Exercise 15.1.3
Show that if $I \subset \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is continuous, then $f(I)$ is an interval.

Exercise 15.1.4
Suppose $f : (a, b) \to \mathbb{R}$ is continuous and strictly monotonic. Let $(c, d) = f((a, b))$. Show that $f^{-1} : (c, d) \to (a, b)$ is strictly monotonic and continuous.

Exercise 15.1.5
Let $n \in \mathbb{Z}^+$. Show that the function $f(x) = \sqrt[n]{x}$ is continuous on $(0, +\infty)$.

Exercise 15.1.6
Use the method of bisection to give another proof of the Intermediate Value Theorem.

15.2 Extreme Value Theorem

Theorem Suppose $D \subset \mathbb{R}$ is compact and $f : D \to \mathbb{R}$ is continuous. Then $f(D)$ is compact.

Proof Given a sequence $\{y_n\}_{n \in I}$ in $f(D)$, choose a sequence $\{x_n\}_{n \in I}$ such that $f(x_n) = y_n$. Since D is compact, $\{x_n\}_{n \in I}$ has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with

$$\lim_{k \to \infty} x_{n_k} = x \in D.$$

Let $y = f(x)$. Then $y \in f(D)$ and, since f is continuous,

$$y = \lim_{k \to \infty} f(x_{n_k}) = \lim_{k \to \infty} y_{n_k}.$$

Hence $f(D)$ is compact.

Exercise 15.2.1
Proof the previous theorem using the open cover definition of a compact set.

The following theorem is known as the *Extreme Value Theorem.*

Theorem Suppose $D \subset \mathbb{R}$ is compact and $f : D \to \mathbb{R}$ is continuous. Then there exists $a \in D$ such that $f(a) \geq f(x)$ for all $x \in D$ and there exists $b \in D$ such that $f(b) \leq f(x)$ for all $x \in D$.

Proof Let $s = \sup f(D)$ and $t = \inf f(D)$. Then $s \in f(D)$, so there exists $a \in D$ such that $f(a) = s$, and $t \in f(D)$, so there exists $b \in D$ such that $f(b) = t$.

As a consequence of the Extreme Value Theorem, a continuous function on a closed bounded interval attains both a maximum and a minimum value.

Exercise 15.2.2
Find an example of a closed bounded interval $[a, b]$ and a function $f : [a, b] \to \mathbb{R}$ such that f attains neither a maximum nor a minimum value on $[a, b]$.

Exercise 15.2.3
Find an example of a bounded interval I and a function $f : I \to \mathbb{R}$ which is continuous on I such that f attains neither a maximum nor a minimum value on I.

Exercise 15.2.4
Suppose $K \subset \mathbb{R}$ is compact and $a \notin K$. Show that there exists $b \in K$ such that $|b - a| \leq |x - a|$ for all $x \in K$.

Proposition Suppose $D \subset \mathbb{R}$ is compact, $f : D \to \mathbb{R}$ is one-to-one, and $E = f(D)$. Then $f^{-1} : E \to D$ is continuous.

Proof Let $V \subset \mathbb{R}$ be an open set. We need to show that $f(V \cap D) = U \cap E$ for some open set $U \subset \mathbb{R}$. Let $C = D \cap (\mathbb{R} \setminus V)$. Then C is a closed subset of D, and so is compact. Hence $f(C)$ is a compact subset of E. Thus $f(C)$ is closed, and so $U = \mathbb{R} \setminus f(C)$ is open. Moreover, $U \cap E = E \setminus f(C) = f(V \cap D)$. Thus f^{-1} is continuous.

Exercise 15.2.5
Suppose $f : [0, 1] \cup (2, 3] \to [0, 2]$ by

$$f(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1, \\ x - 1, & \text{if } 2 < x \leq 3. \end{cases}$$

Show that f is continuous, one-to-one, and onto, but that f^{-1} is not continuous.

15.3 Uniform continuity

Definition Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. We say f is *uniformly continuous* on D if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any $x, y \in D$,

$$|f(x) - f(y)| < \epsilon$$

whenever $|x - y| < \delta$.

Exercise 15.3.1
Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ is Lipschitz. Show that f is uniformly continuous on D.

Clearly, if f is uniformly continuous on D then f is continuous on D. However, a continuous function need not be uniformly continuous.

Example Define $f : (0, +\infty)$ by $f(x) = \frac{1}{x}$. Given any $\delta > 0$, choose $n \in \mathbb{Z}^+$ such that $\frac{1}{n(n+1)} < \delta$. Let $x = \frac{1}{n}$ and $y = \frac{1}{n+1}$. Then

$$|x - y| = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} < \delta.$$

However,

$$|f(x) - f(y)| = |n - (n+1)| = 1.$$

Hence, for example, there does not exist a $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{1}{2}$$

whenever $|x - y| < \delta$. Thus f is not uniformly continuous on $(0, +\infty)$, although f is continuous on $(0, +\infty)$.

Example Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x$. Let $\epsilon > 0$ be given. If $\delta = \frac{\epsilon}{2}$, then

$$|f(x) - f(y)| = 2|x - y| < \epsilon$$

whenever $|x - y| < \delta$. Hence f is uniformly continuous on $\mathbb{R}$.

Exercise 15.3.2
Show that $f(x) = x^2$ is not uniformly continuous on $(-\infty, +\infty)$.

Proposition Suppose $D \subset \mathbb{R}$ is compact and $f : D \to \mathbb{R}$ is continuous. Then f is uniformly continuous on D.

Proof Let $\epsilon > 0$ be given. For every $x \in D$, choose δ_x such that

$$|f(x) - f(y)| < \frac{\epsilon}{2}$$

whenever $y \in D$ and $|x - y| < \delta_x$. Let

$$J_x = (x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2}).$$

Then $\{J_x : x \in D\}$ is an open cover of D. Since D is compact, there must exist $x_1, x_2, \ldots, x_n$, $n \in Z^+$, such that $J_{x_1}, J_{x_2}, \ldots, J_{x_n}$ is an open cover of D. Let δ be the smallest of

$$\frac{\delta_{x_1}}{2}, \frac{\delta_{x_2}}{2}, \ldots, \frac{\delta_{x_n}}{2}.$$

Now let $x, y \in D$ with $|x - y| < \delta$. Then for some integer k, $1 \le k \le n$, $x \in J_{x_k}$, that is,

$$|x - x_k| < \frac{\delta_{x_k}}{2}.$$

Moreover,

$$|y - x_k| \leq |y - x| + |x - x_k| < \delta + \frac{\delta_{x_k}}{2} \leq \delta_{x_k}.$$

Thus

$$|f(x) - f(y)| \leq |f(x) - f(x_k)| + |f(x_k) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Exercise 15.3.3
Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ is uniformly continuous. Show that if $\{x_n\}_{n \in I}$ is a Cauchy sequence in D, then $\{f(x_n)\}_{n \in I}$ is a Cauchy sequence in $f(D)$.

Exercise 15.3.4
Suppose $f : (0, 1) \to \mathbb{R}$ is uniformly continuous. Show that $f(0+)$ exists.

Exercise 15.3.5
Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to +\infty} f(x) = 0$. Show that f is uniformly continuous.