Lecture 14: Continuous Functions

14.1 Continuity at a point

Definition Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $a \in D$. We say f is *continuous* at a if either a is an isolated point of D or $\lim_{x \to a} f(x) = f(a)$. If f is not continuous at a, we say f is *discontinuous* at a, or that f has a *discontinuity* at a.

Example Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Then f is discontinuous at every $x \in \mathbb{R}$.

Example Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Then f is continuous at 0, but discontinuous at every $x \neq 0$.

If $D \subset \mathbb{R}$, $\alpha \in \mathbb{R}$, $f : D \to \mathbb{R}$, and $g : D \to \mathbb{R}$, then we define $\alpha f : D \to \mathbb{R}$ by $(\alpha f)(x) = \alpha f(x)$, $f + g : D \to \mathbb{R}$ by $(f + g)(x) = f(x) + g(x)$, and $fg : D \to \mathbb{R}$ by $(fg)(x) = f(x)g(x)$. Moreover, if $g(x) \neq 0$ for all $x \in D$, we define $\frac{f}{g} : D \to \mathbb{R}$ by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

Proposition Suppose $D \subset \mathbb{R}$, $\alpha \in \mathbb{R}$, $f : D \to \mathbb{R}$, and $g : D \to \mathbb{R}$. If f and g are continuous at a, then αf, $f + g$, and fg are all continuous at a. Moreover, if $g(x) \neq 0$ for all $x \in D$, then $\frac{f}{g}$ is continuous at a.

Exercise 14.1.1
Prove the previous proposition.

Proposition Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, $f(x) \geq 0$ for all $x \in D$, and f is continuous at $a \in D$. If $g : D \to \mathbb{R}$ is defined by $g(x) = \sqrt{f(x)}$, then g is continuous at a.

Exercise 14.1.2
Prove the previous proposition.

Proposition Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $a \in D$. Then f is continuous at a if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon$$

whenever

$$x \in (a - \delta, a + \delta) \cap D.$$

Proof Suppose f is continuous at a. If a is an isolated point of D, then there exists a $\delta > 0$ such that

$$(a - \delta, a + \delta) \cap D = \{a\}.$$

Then for any $\epsilon > 0$,

$$|f(x) - f(a)| = |f(a) - f(a)| = 0 < \epsilon$$

whenever

$$x \in (a - \delta, a + \delta) \cap D.$$

If a is a limit point of D, then $\lim_{x \to a} f(x) = f(a)$ implies that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon$$

whenever

$$x \in (a - \delta, a + \delta) \cap D.$$

Now suppose that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon$$

whenever

$$x \in (a - \delta, a + \delta) \cap D.$$

If a is an isolated point, then f is continuous at a. If a is a limit point, then this condition implies $\lim_{x \to a} f(x) = f(a)$, and so f is continuous at a.

From the preceding, it should be clear that a function $f : D \to \mathbb{R}$ is continuous at a point a of D if and only if for every sequence $\{x_n\}_{n \in I}$ with $x_n \in D$ for every $n \in I$ and $\lim_{n \to \infty} x_n = a$, $\lim_{n \to \infty} f(x_n) = f(a)$.

Exercise 14.1.3
Show that if $f : D \to \mathbb{R}$ is continuous at $a \in D$ and $f(a) > 0$, then there exists an open interval I such that $a \in I$ and $f(x) > 0$ for every $x \in I \cap D$.

Proposition Suppose $D \subset \mathbb{R}$, $E \subset \mathbb{R}$, $g : D \to \mathbb{R}$, $f : E \to \mathbb{R}$, $g(D) \subset E$, and $a \in D$. If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

Proof Let $\{x_n\}_{n \in I}$ be a sequence with $x_n \in D$ for every $n \in I$ and $\lim_{n \to \infty} x_n = a$. Then, since g is continuous at a, $\{g(x_n)\}_{n \in I}$ is a sequence with $g(x_n) \in E$ for every $n \in I$ and $\lim_{n \to \infty} g(x_n) = g(a)$. Hence, since f is continuous at $g(a)$, $\lim_{n \to \infty} f(g(x_n)) = f(g(a))$. That is, $\lim_{n \to \infty} (f \circ g)(x_n) = (f \circ g)(a)$. Hence $f \circ g$ is continuous at a.

Definition Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $a \in D$. If f is not continuous at a but both $f(a-)$ and $f(a+)$ exist, then we say f has a *simple discontinuity* at a.

Proposition Suppose f is monotonic on the interval (a, b). Then every discontinuity of f in (a, b) is a simple discontinuity. Moreover, if E is the set of points in (a, b) at which f is discontinuous, then either $E = \emptyset$, E is finite, or E is countable.

Proof The first statement follows immediately from a previous result. For the second statement, suppose f is nondecreasing and suppose E is nonempty. From our previous result we know that for every $x \in (a, b)$,

$$f(x-) \leq f(x) \leq f(x+).$$

Hence $x \in E$ if and only if $f(x-) < f(x+)$. Hence for every $x \in E$, we may choose a rational number r_x such that $f(x-) < r_x < f(x+)$. Now if $x, y \in E$ with $x < y$, then

$$r_x < f(x+) \leq f(y-) < r_y,$$

so $r_x \neq r_y$. Thus we have a one-to-one correspondence between E and a subset of $\mathbb{Q}$, and so E is either finite or countable. A similar argument holds if f is nonincreasing.

Exercise 14.1.4
Define $f : \mathbb{R} \to \mathbb{R}$ define by

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x \text{ is rational and } x = \frac{p}{q}, \\ 0, & \text{if } x \text{ is irrational,} \end{cases}$$

where p and q are taken to be relatively prime integers with $q > 0$, and we take $q = 1$ when $x = 0$. Show that f is continuous at every irrational number and has simple discontinuity at every rational number.

14.2 Continuity on a set

Definition Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. We say f is *continuous on* D if f is continuous at every point $a \in D$.

Proposition If f is a polynomial, then f is continuous on $\mathbb{R}$.

Proposition If $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ is a rational function, then f is continuous on D.

Exercise 14.2.1
Explain why the function $f(x) = \sqrt{1 - x^2}$ is continuous on $[-1, 1]$.

Exercise 14.2.2
Discuss the continuity of the function

$$f(x) = \begin{cases} x + 1, & \text{if } x < 0, \\ 4, & \text{if } x = 0, \\ x^2, & \text{if } x > 0. \end{cases}$$

If $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $E \subset \mathbb{R}$, we let

$$f^{-1}(E) = \{x : f(x) \in E\}.$$

Proposition Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Then f is continuous on D if and only if for every open set $V \subset \mathbb{R}$, $f^{-1}(V) = U \cap D$ for some open set $U \subset \mathbb{R}$.

Proof Suppose f is continuous on D and $V \subset \mathbb{R}$ is an open set. If $V \cap f(D) = \emptyset$, then $f^{-1}(V) = \emptyset$, which is open. So suppose $V \cap f(D) \neq \emptyset$ and let $a \in f^{-1}(V)$. Since V is open and $f(a) \in V$, there exists $\epsilon_a > 0$ such that

$$(f(a) - \epsilon_a, f(a) + \epsilon_a) \subset V.$$

Since f is continuous, there exists $\delta_a > 0$ such that

$$f((a - \delta_a, a + \delta_a) \cap D) \subset (f(a) - \epsilon_a, f(a) + \epsilon_a) \subset V.$$

That is, $(a - \delta_a, a + \delta_a) \cap D \subset f^{-1}(V)$. Let

$$U = \bigcup_{a \in f^{-1}(V)} (a - \delta_a, a + \delta_a).$$

Then U is open and $f^{-1}(V) = U \cap D$.

Now suppose that for every open set $V \subset \mathbb{R}$, $f^{-1}(V) = U \cap D$ for some open set $U \subset \mathbb{R}$. Let $a \in D$ and let $\epsilon > 0$ be given. Since $(f(a) - \epsilon, f(a) + \epsilon)$ is open, there exists an open set U such that

$$U \cap D = f^{-1}((f(a) - \epsilon, f(a) + \epsilon)).$$

Since U is open and $a \in U$, there exists $\delta > 0$ such that $(a - \delta, a + \delta) \subset U$. But then

$$f((a - \delta, a + \delta) \cap D) \subset (f(a) - \epsilon, f(a) + \epsilon).$$

That is, if $x \in (a - \delta, a + \delta) \cap D$, then $|f(x) - f(a)| < \epsilon$. Hence f is continuous at a.

Exercise 14.2.3
Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. For any $E \subset \mathbb{R}$, show that $f^{-1}(\mathbb{R} \setminus E) = (\mathbb{R} \setminus f^{-1}(E)) \cap D$.

Exercise 14.2.4
Let A be a set and, for each $\alpha \in A$, let $U_\alpha \subset \mathbb{R}$. Given $D \subset \mathbb{R}$ and a function $f : D \to \mathbb{R}$, show that

$$\bigcup_{\alpha \in A} f^{-1}(U_\alpha) = f^{-1}\left(\bigcup_{\alpha \in A} U_\alpha\right)$$

and

$$\bigcap_{\alpha \in A} f^{-1}(U_\alpha) = f^{-1}\left(\bigcap_{\alpha \in A} U_\alpha\right).$$

Exercise 14.2.5
Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Show that f is continuous on D if and only if for every closed set $C \subset \mathbb{R}$, $f^{-1}(C) = F \cap D$ for some closed set $F \subset \mathbb{R}$.

Exercise 14.2.6
Let $D \subset \mathbb{R}$. A function $f : D \to \mathbb{R}$ is said to be *Lipschitz* if there exists $\alpha \in \mathbb{R}$, $\alpha > 0$, such that $|f(x) - f(y)| \leq \alpha |x - y|$ for all $x, y \in D$. Show that if f is Lipschitz, then f is continuous.