Lecture 13: More on Limits of Functions

13.1 Equivalent forms of the definitions

Proposition Suppose $D \subset \mathbb{R}$, a is a limit point of D, and $f : D \to \mathbb{R}$. Then $\lim_{x\to a} f(x) = L$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon$$

whenever $x \neq a$ and

$$x \in (a - \delta, a + \delta) \cap D.$$

Proof Suppose $\lim_{x\to a} f(x) = L$. Suppose there exists an $\epsilon > 0$ such that for every $\delta > 0$ there exists $x \in (a-\delta, a+\delta) \cap D$, $x \neq a$, for which $|f(x) - L| \geq \epsilon$. For $n = 1, 2, 3, \ldots$, choose

$$x_n \in (a - \frac{1}{n}, a + \frac{1}{n}) \cap D,$$

$x_n \neq a$, such that $|f(x_n) - L| \geq \epsilon$. Then $\{x_n\}_{n=1}^{\infty} \in S(D, a)$, but $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to L, contradicting the assumption that $\lim_{x\to a} f(x) = L$.

Now suppose that for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $x \neq a$ and $x \in (a - \delta, a + \delta) \cap D$. Let $\{x_n\}_{n\in I} \in S(D, a)$. Given $\epsilon > 0$, let $\delta > 0$ be such that $|f(x) - L| < \epsilon$ whenever $x \neq a$ and $x \in (a - \delta, a + \delta) \cap D$. Choose $N \in \mathbb{Z}$ such that $|x_n - a| < \delta$ whenever $n > N$. It follows that $|f(x_n) - L| < \epsilon$ for all $n > N$. Hence $\lim_{n\to\infty} f(x_n) = L$, and so $\lim_{x\to a} f(x) = L$.

The following two propositions are proven analogously.

Proposition Suppose $D \subset \mathbb{R}$, a is a limit point of D, $f : D \to \mathbb{R}$, and $S^-(D, a) \neq \emptyset$. Then $\lim_{x\to a^-} f(x) = L$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon$$

whenever

$$x \in (a - \delta, a) \cap D.$$

Proposition Suppose $D \subset \mathbb{R}$, a is a limit point of D, $f : D \to \mathbb{R}$, and $S^+(D, a) \neq \emptyset$. Then $\lim_{x\to a^+} f(x) = L$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon$$

whenever

$$x \in (a, a + \delta) \cap D.$$

13.2 Examples

Example Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Let $a \in \mathbb{R}$. Since every open interval contains both rational and irrational numbers, for any $\delta > 0$ and any choice of $L \in \mathbb{R}$, there will exist $x \in (a - \delta, a + \delta)$, $x \neq a$, such that

$$|f(x) - L| \geq \frac{1}{2}.$$

Hence $\lim_{x \to a} f(x)$ does not exist for any real number a.

Example Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Then $\lim_{x \to 0} f(x) = 0$ since, given $\epsilon > 0$, $|f(x)| < \epsilon$ provided $|x| < \epsilon$.

Exercise 13.2.1
Show that if f is as given in the previous example and $a \neq 0$, then $\lim_{x \to a} f(x)$ does not exist.

Exercise 13.2.2
Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x \text{ is rational and } x = \frac{p}{q}, \\ 0, & \text{if } x \text{ is irrational,} \end{cases}$$

where p and q are taken to be relatively prime integers with $q > 0$, and we take $q = 1$ when $x = 0$. Show that, for any real number a, $\lim_{x \to a} f(x) = 0$.

Example Define $\varphi : [0, 1] \to [-1, 1]$ by

$$\varphi(x) = \begin{cases} 4x, & \text{if } 0 \leq x \leq \frac{1}{4}, \\ 2 - 4x, & \text{if } \frac{1}{4} < x < \frac{3}{4}, \\ 4x - 4, & \text{if } \frac{3}{4} \leq x \leq 1. \end{cases}$$

Next define $s : \mathbb{R} \to \mathbb{R}$ by $s(x) = \varphi(x - \lfloor x \rfloor)$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x (that is, $\lfloor x \rfloor$ is the *floor* of x). The function s is an example of a *sawtooth* function. Note that for any $n \in \mathbb{Z}$,

$$s([n, n + 1]) = [-1, 1].$$

Finally, let $D = \mathbb{R} \setminus \{0\}$ and define $f : D \to \mathbb{R}$ by

$$f(x) = s\left(\frac{1}{x}\right).$$

Note that for any $n \in \mathbb{Z}^+$,

$$f\left(\left[\frac{1}{n+1}, \frac{1}{n}\right]\right) = s([n, n+1]) = [-1, 1].$$

Hence for any $\epsilon > 0$, $f((0,\epsilon)) = [-1,1]$, and so $\lim_{x\to 0^+} f(x)$ does not exist. Similarly, neither $\lim_{x\to 0^-} f(x)$ nor $\lim_{x\to 0} f(x)$ exist.

Example Let s be the sawtooth function of the previous example and let $D = \mathbb{R} \setminus \{0\}$. Define $f : D \to \mathbb{R}$ by

$$f(x) = xs\left(\frac{1}{x}\right).$$

Then for all $x \in D$,

$$-|x| \leq f(x) \leq |x|,$$

and so $\lim_{x\to 0} f(x) = 0$ by the squeeze theorem.

Definition Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. We say f is *bounded* if there exists a real number B such that $|f(x)| \leq B$ for all $x \in D$.

Exercise 13.2.3
Suppose $f : \mathbb{R} \to \mathbb{R}$ is bounded. Show that $\lim_{x\to 0} xf(x) = 0$.

13.3 Monotonic functions

Definition Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $(a,b) \subset D$. We say f is *increasing* on (a,b) if $f(x) < f(y)$ whenever $a < x < y < b$; we say f is *decreasing* on (a,b) if $f(x) > f(y)$ whenever $a < x < y < b$; we say f is *nondecreasing* on (a,b) if $f(x) \leq f(y)$ whenever $a < x < y < b$; and we say f is *nonincreasing* on (a,b) if $f(x) \geq f(y)$ whenever $a < x < y < b$. We will say f is *monotonic* on (a,b) if f is either nondecreasing or nonincreasing on (a,b) and we will say f is *strictly monotonic* on (a,b) if f is either increasing or decreasing on (a,b).

Proposition If f is monotonic on (a,b), then $f(c+)$ and $f(c-)$ exist for every $c \in (a,b)$.

Proof Suppose f is nondecreasing on (a,b). Let $c \in (a,b)$. Let

$$\lambda = \sup\{f(x) : a < x < c\}.$$

Note that $\lambda \leq f(c) < +\infty$. Given any $\epsilon > 0$, there must exist $\delta > 0$ such that

$$\lambda - \epsilon < f(c - \delta) \leq \lambda.$$

Since f is nondecreasing, it follows that

$$|f(x)-\lambda|<\epsilon$$

whenever $x \in (c-\delta, c)$. Thus $f(c-)=\lambda$. A similar argument shows that $f(c+)=\kappa$ where

$$\kappa=\inf\{f(x): c<x<b\}.$$

If f is nonincreasing, similar arguments yield

$$f(c-)=\inf\{f(x): a<x<c\}$$

and

$$f(c+)=\sup\{f(x): c<x<b\}.$$

Proposition If f is nondecreasing on (a, b) and $a<x<y<b$, then

$$f(x+) \leq f(y-).$$

Proof By the previous proposition,

$$f(x+)=\inf\{f(t): x<t<b\}$$

and

$$f(y-)=\sup\{f(t): a<t<y\}.$$

Since f is nondecreasing,

$$\inf\{f(t): x<t<b\}=\inf\{f(t): x<t<y\}$$

and

$$\sup\{f(t): a<t<y\}=\sup\{f(t): x<t<y\}.$$

Thus

$$f(x+)=\inf\{f(t): x<t<y\} \leq \sup\{f(t): x<t<y\}=f(y-).$$

Exercise 13.3.1

Let $\varphi: \mathbb{Q} \cap[0,1] \rightarrow \mathbb{Z}^{+}$ be a one-to-one correspondence. Define $f:[0,1] \rightarrow \mathbb{R}$ by

$$f(x)=\sum_{\substack{q \in \mathbb{Q} \cap[0,1] \\ q \leq x}} \frac{1}{2^{\varphi(q)}}.$$

(a) Show that f is increasing on $(0,1)$.
(b) Show that for any $x \in \mathbb{Q} \cap(0,1)$, $f(x-)<f(x)$ and $f(x+)=f(x)$.
(c) Show that for any irrational a, $0<a<1$, $\lim_{x \rightarrow a} f(x)=f(a)$.

13.4 Limits to infinity and infinite limits

Definition Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and suppose a is a limit point of D. We say that that f *diverges* to $+\infty$ as x approaches a, denoted

$$\lim_{x \to a} f(x) = +\infty,$$

if for every real number M there exists a $\delta > 0$ such that

$$f(x) > M$$

whenever $x \neq a$ and

$$x \in (a - \delta, a + \delta) \cap D.$$

Similarly, we say that that f *diverges* to $-\infty$ as x approaches a, denoted

$$\lim_{x \to a} f(x) = -\infty,$$

if for every real number M there exists a $\delta > 0$ such that

$$f(x) < M$$

whenever $x \neq a$ and

$$x \in (a - \delta, a + \delta) \cap D.$$

Exercise 13.4.1
Write definitions for $\lim_{x \to a^+} f(x) = +\infty$, $\lim_{x \to a^-} f(x) = +\infty$, $\lim_{x \to a^+} f(x) = -\infty$, and $\lim_{x \to a^-} f(x) = -\infty$. Model your definitions on the preceding definitions.

Exercise 13.4.2
Show that $\displaystyle\lim_{x \to 4^+} \frac{7}{4 - x} = -\infty$ and $\displaystyle\lim_{x \to 4^-} \frac{7}{4 - x} = +\infty$

Definition Suppose $D \subset \mathbb{R}$ does not have an upper bound, $f : D \to \mathbb{R}$, and $L \in \mathbb{R}$. We say that the *limit* of f as x approaches $+\infty$ is L, denoted

$$\lim_{x \to +\infty} f(x) = L$$

if for every $\epsilon > 0$ there exists a real number M such that

$$|f(x) - L| < \epsilon$$

whenever $x \in (M, +\infty) \cap D$.

Definition Suppose $D \subset \mathbb{R}$ does not have an lower bound, $f : D \to \mathbb{R}$, and $L \in \mathbb{R}$. We say that the *limit* of f as x approaches $-\infty$ is L, denoted

$$\lim_{x \to -\infty} f(x) = L$$

if for every $\epsilon > 0$ there exists a real number M such that

$$|f(x) - L| < \epsilon$$

whenever $x \in (-\infty, M) \cap D$.

Exercise 13.4.3
Verify that $\displaystyle\lim_{x \to +\infty} \frac{x+1}{x+2} = 1$.

Exercise 13.4.4
Provide definitions for $\lim_{x\to+\infty} f(x) = +\infty$, $\lim_{x\to+\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = +\infty$, and $\lim_{x\to-\infty} f(x) = -\infty$. Model your definitions on the preceding definitions.

Exercise 13.4.5
Suppose

$$f(x) = ax^3 + bx^2 + cx + d,$$

where $a, b, c, d \in \mathbb{R}$ and $a > 0$. Show that $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.