Lecture 12: Limits of Functions

12.1 Limits

Let $A \subset \mathbb{R}$ and let x be a limit point of A. In the following, it will be convenient to let $S(A, x)$ denote the set of all convergent sequences $\{x_n\}_{n \in I}$ such that $x_n \in A$ for all $n \in I$, $x_n \neq x$ for all $n \in I$, and $\lim_{n \to \infty} x_n = x$. We will let $S^+(A, x)$ be the subset of $S(A, x)$ of sequences $\{x_n\}_{n \in I}$ for which $x_n > x$ for all $n \in I$ and $S^-(A, x)$ be the subset of $S(A, x)$ of sequences $\{x_n\}_{n \in I}$ for which $x_n < x$ for all $n \in I$.

Definition Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, $L \in \mathbb{R}$, and suppose a is a limit point of D. We say the *limit* of f as x approaches a is L, denoted

$$\lim_{x \to a} f(x) = L,$$

if for every sequence $\{x_n\}_{n \in I} \in S(D, a)$,

$$\lim_{n \to \infty} f(x_n) = L.$$

If $S^+(D, a) \neq \emptyset$, we say the *limit from the right* of f as x approaches a is L, denoted

$$\lim_{x \to a^+} f(x) = L,$$

if for every sequence $\{x_n\}_{n \in I} \in S^+(D, a)$,

$$\lim_{n \to \infty} f(x_n) = L,$$

and, if $S^-(D, a) \neq \emptyset$, we say the *limit from the left* of f as x approaches a is L, denoted

$$\lim_{x \to a^-} f(x) = L,$$

if for every sequence $\{x_n\}_{n \in I} \in S^-(D, a)$,

$$\lim_{n \to \infty} f(x_n) = L.$$

We may also denote $\lim_{x \to a} f(x) = L$ by writing $f(x) \to L$ as $x \to a$. We also let

$$f(a+) = \lim_{x \to a^+} f(x)$$

and

$$f(a-) = \lim_{x \to a^-} f(x).$$

It should be clear that if $\lim_{x \to a} f(x) = L$ and $S^+(D, a) \neq \emptyset$, then $f(a+) = L$. Similarly, if $\lim_{x \to a} f(x) = L$ and $S^-(D, a) \neq \emptyset$, then $f(a-) = L$.

Proposition Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and a is a limit point of D. If $f(a-) = f(a+) = L$, then $\lim_{x \to a} f(x) = L$.

Proof Suppose $\{x_n\}_{n=m}^{\infty} \in S(D, a)$. Let

$$J^- = \{n : n \in \mathbb{Z}, x_n < a\}$$

and

$$J^+ = \{n : n \in \mathbb{Z}, x_n > a\}.$$

Suppose J^- is empty or finite and let $k = m - 1$ if $J^- = \emptyset$ and, otherwise, let k be the largest integer in J^-. Then $\{x_n\}_{n=k+1}^{\infty} \in S^+(D, a)$, and so

$$\lim_{n \to \infty} f(x_n) = f(a+) = L.$$

A similar argument shows that if J^+ is empty or finite, then

$$\lim_{n \to \infty} f(x_n) = f(a-) = L.$$

If neither J^- nor J^+ is finite or empty, then $\{x_n\}_{n \in J^-}$ and $\{x_n\}_{n \in J^+}$ are subsequences of $\{x_n\}_{n=m}^{\infty}$ with $\{x_n\}_{n \in J^-} \in S^-(D, a)$ and $\{x_n\}_{n \in J^+} \in S^+(D, a)$. Hence, given any $\epsilon > 0$, we may find integers N and M such that

$$|f(x_n) - L| < \epsilon$$

whenever $n \in \{j : j \in J^-, j > N\}$ and

$$|f(x_n) - L| < \epsilon$$

whenever $n \in \{j : j \in J^+, j > M\}$. Let P be the larger of N and M. Since $J^- \cup J^+ = \{j : j \in \mathbb{Z}^+, j \geq m\}$, it follows that

$$|f(x_n) - L| < \epsilon$$

whenever $n > P$. Hence $\lim_{n \to \infty} f(x_n) = L$, and so $\lim_{x \to a} f(x) = L$.

Proposition Suppose $D \subset \mathbb{R}$, a is a limit point of D, and $f : D \to \mathbb{R}$. If $\lim_{x \to a} f(x) = L$ and $\alpha \in \mathbb{R}$, then

$$\lim_{x \to a} \alpha f(x) = \alpha L.$$

Proof Suppose $\{x_n\}_{n \in I} \in S(D, a)$. Then

$$\lim_{n \to \infty} \alpha f(x_n) = \alpha \lim_{n \to \infty} f(x_n) = \alpha L.$$

Hence $\lim_{x \to a} \alpha f(x) = \alpha L$.

Proposition Suppose $D \subset \mathbb{R}$, a is a limit point of D, $f : D \to \mathbb{R}$, and $g : D \to \mathbb{R}$. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

$$\lim_{x\to a}(f(x) + g(x)) = L + M.$$

Proof Suppose $\{x_n\}_{n\in I} \in S(D, a)$. Then

$$\lim_{n\to\infty}(f(x_n) + g(x_n)) = \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} g(x_n) = L + M.$$

Hence $\lim_{x\to a}(f(x) + g(x)) = L + M$.

Proposition Suppose $D \subset \mathbb{R}$, a is a limit point of D, $f : D \to \mathbb{R}$, and $g : D \to \mathbb{R}$. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

$$\lim_{x\to a} f(x)g(x) = LM.$$

Exercise 12.1.1
Prove the previous proposition.

Proposition Suppose $D \subset \mathbb{R}$, a is a limit point of D, $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$, and $g(x) \neq 0$ for all $x \in D$. If $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$, and $M \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Exercise 12.1.2
Prove the previous proposition.

Proposition Suppose $D \subset \mathbb{R}$, a is a limit point of D, $f : D \to \mathbb{R}$, and $f(x) \geq 0$ for all $x \in D$. If $\lim_{x\to a} f(x) = L$, then

$$\lim_{x\to a} \sqrt{f(x)} = \sqrt{L}.$$

Exercise 12.1.3
Prove the previous proposition.

Given $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $A \subset D$, we let

$$f(A) = \{y : y = f(x) \text{ for some } x \in A\}.$$

In particular, $f(D)$ denotes the range of f.

Proposition Suppose $D \subset \mathbb{R}$, $E \subset \mathbb{R}$, a is a limit point of D, $g : D \to \mathbb{R}$, $f : E \to \mathbb{R}$, and $g(D) \subset E$. Moreover, suppose $\lim_{x\to a} g(x) = b$ and, for some $\epsilon > 0$, $g(x) \neq b$ for all $x \in (a - \epsilon, a + \epsilon) \cap D$. If $\lim_{x\to b} f(x) = L$, then

$$\lim_{x\to a} f \circ g(x) = L.$$

Proof Suppose $\{x_n\}_{n \in I} \in S(D, a)$. Then

$$\lim_{n \to \infty} g(x_n) = b.$$

Let $N \in \mathbb{Z}^+$ such that $|x_n - a| < \epsilon$ whenever $n > N$. Then

$$\{g(x_n)\}_{n=N+1}^{\infty} \in S(E, b),$$

so

$$\lim_{n \to \infty} f(g(x_n)) = L.$$

Thus $\lim_{x \to a} f \circ g(x) = L$.

Example Let

$$g(x) = \begin{cases} 0, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

If $f(x) = g(x)$, then

$$f \circ g(x) = \begin{cases} 1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Hence $\lim_{x \to 0} f \circ g(x) = 1$, although $\lim_{x \to 0} g(x) = 0$ and $\lim_{x \to 0} f(x) = 0$.

12.2 Important examples of limits

Example If $c \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = c$ for all $x \in \mathbb{R}$, then clearly $\lim_{x \to a} f(x) = c$ for any $a \in \mathbb{R}$.

Example Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x$ for all $x \in \mathbb{R}$. If, for any $a \in \mathbb{R}$, $\{x_n\}_{n \in I} \in S(\mathbb{R}, a)$, then

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_n = a.$$

Hence $\lim_{x \to a} x = a$.

Example Suppose $n \in \mathbb{Z}^+$ and $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^n$. Then

$$\lim_{x \to a} f(x) = \lim_{x \to a} x^n = \prod_{i=1}^{n} \lim_{x \to a} x = a^n.$$

Definition If $n \in \mathbb{Z}$, $n \geq 0$, and $b_0, b_1, \ldots, b_n$ are real numbers with $b_n \neq 0$, then the function $p : \mathbb{R} \to \mathbb{R}$ defined by

$$p(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$$

is called a *polynomial* of *degree* n.

Exercise 12.2.1
Show that if f is a polynomial and $a \in \mathbb{R}$, then $\lim_{x \to a} f(x) = f(a)$.

Definition Suppose p and q are polynomials and $D = \{x : x \in \mathbb{R}, q(x) \neq 0\}$. The function $r : D \to \mathbb{R}$ defined by

$$r(x) = \frac{p(x)}{q(x)}$$

is called a *rational function*.

Exercise 12.2.2
Show that if f is a rational function and a is in the domain of f, then $\lim_{x \to a} f(x) = f(a)$.

Exercise 12.2.3
Suppose $D \subset \mathbb{R}$, $a \in D$ is a limit point of D, and $\lim_{x \to a} f(x) = L$. If $E = D \setminus \{a\}$ and $g : E \to \mathbb{R}$ is defined by $g(x) = f(x)$ for all $x \in E$, show that $\lim_{x \to a} g(x) = L$.

Exercise 12.2.4
Evaluate

$$\lim_{x \to 1} \frac{x^5 - 1}{x^3 - 1}.$$

Exercise 12.2.5
Suppose $D \subset \mathbb{R}$, a is a limit point of D, $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$, $h : D \to \mathbb{R}$, and $f(x) \leq h(x) \leq g(x)$ for all $x \in D$. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = L$, show that $\lim_{x \to a} h(x) = L$. (This is the *squeeze theorem* for limits of functions.)

Note that the results of this lecture which have been stated for limits will hold as well for the appropriate one-sided limits, that is, limits from the right or from the left.

Exercise 12.2.6
Suppose

$$f(x) = \begin{cases} x + 1, & \text{if } x < 0, \\ 4, & \text{if } x = 0, \\ x^2, & \text{if } x > 0. \end{cases}$$

Evaluate $f(0)$, $f(0-)$, and $f(0+)$. Does $\lim_{x \to 0} f(x)$ exist?