Lecture 11: Compact Sets

11.1 Compact Sets

Definition Suppose $T \subset \mathbb{R}$. If A is a set, U_α is an open set for every $\alpha \in A$, and

$$T \subset \bigcup_{\alpha \in A} U_\alpha,$$

then we call $\{U_\alpha : \alpha \in A\}$ an *open cover* of T.

Example For $n = 3, 4, 5, \ldots$, let

$$U_n = \left(\frac{1}{n}, \frac{n-1}{n}\right).$$

Then $\{U_n : i = 3, 4, 5, \ldots\}$ is an open cover of the open interval $(0, 1)$.

Definition Suppose $\{U_\alpha : \alpha \in A\}$ is an open cover of $T \subset \mathbb{R}$. If $B \subset A$ and

$$T \subset \bigcup_{\beta \in B} U_\beta,$$

then we call $\{U_\beta : \beta \in B\}$ a *subcover* of $\{U_\alpha : \alpha \in A\}$. If B is finite, we call $\{U_\beta : \beta \in B\}$ a *finite subcover* of $\{U_\alpha : \alpha \in A\}$

Exercise 11.1.1
Show that the open cover of $(0, 1)$ given in the previous example does not have a finite subcover.

Definition We say a set $K \subset \mathbb{R}$ is *compact* if every open cover of K has a finite subcover.

Example As a consequence of the previous exercise, the open interval $(0, 1)$ is not compact.

Exercise 11.1.2
Show that every finite subset of $\mathbb{R}$ is compact.

Exercise 11.1.3
Suppose $n \in \mathbb{Z}^+$ and $K_1, K_2, \ldots, K_n$ are compact sets. Show that $\bigcup_{i=1}^n K_i$ and $\bigcap_{k=1}^n K_i$ are compact.

Proposition If I is a closed, bounded interval, then I is compact.

Proof Let $a \leq b$ be finite real numbers and $I = [a, b]$. Suppose $\{U_\alpha : \alpha \in A\}$ is an open cover of I. Let $\mathcal{O}$ be the set of sets $\{U_\beta : \beta \in B\}$ with the properties that B is a finite subset of A and $a \in \bigcup_{\beta \in B} U_\beta$. Let

$$T = \{x : x \in I, [a, x] \subset \bigcup_{\beta \in B} U_\beta \text{ for some } \{U_\beta : \beta \in B\} \in \mathcal{O}\}.$$

Clearly, $a \in T$ so $T \neq \emptyset$. Let $s = \sup T$. Suppose $s < b$. Since $\{U_\alpha : \alpha \in A\}$ is an open cover of I, there exists an $\alpha \in A$ for which $s \in U_\alpha$. Hence there exists an $\epsilon > 0$ such that

$$(s - \epsilon, s + \epsilon) \subset U_\alpha.$$

Moreover, there exists a $\{U_\beta : \beta \in B\} \in \mathcal{O}$ for which

$$[a, s - \frac{\epsilon}{2}] \subset \bigcup_{\beta \in B} U_\beta.$$

But then

$$\{U_\beta : \beta \in B\} \cup \{U_\alpha\} \in \mathcal{O}$$

and

$$\left[a, s + \frac{\epsilon}{2}\right] \subset \left(\bigcup_{\beta \in B} U_\beta \right) \cup U_\alpha,$$

contradicting the definition of s. Hence we must have $s = b$. Now choose U_α such that $b \in U_\alpha$. Then, for some $\epsilon > 0$,

$$(b - \epsilon, b + \epsilon) \subset U_\alpha$$

Moreover, there exists $\{U_\beta : \beta \in B\} \in \mathcal{O}$ such that

$$[a, b - \frac{\epsilon}{2}] \subset \bigcup_{\beta \in B} U_\beta.$$

Then

$$\{U_\beta : \beta \in B\} \cup \{U_\alpha\} \in \mathcal{O}$$

is a finite subcover of I. Thus I is compact.

Proposition If K is a closed, bounded subset of $\mathbb{R}$, then K is compact.

Proof Since K is bounded, there exist finite real numbers a and b such that $K \subset [a, b]$. Let $\{U_\alpha : \alpha \in A\}$ be an open cover of K. Let $V = \mathbb{R} \setminus K$. Then

$$\{U_\alpha : \alpha \in A\} \cup \{V\}$$

is an open cover of $[a, b]$. Since $[a, b]$ is compact, there exists a finite subcover of this cover. This subcover is either of the form $\{U_\beta : \beta \in B\}$ or $\{U_\beta : \beta \in B\} \cup \{V\}$ for some $B \subset A$. In the former case, we have

$$K \subset [a, b] \subset \bigcup_{\beta \in B} U_\beta;$$

in the latter case, we have

$$K \subset [a, b] \setminus V \subset \bigcup_{\beta \in B} U_\beta.$$

In either case, we have found a finite subcover of $\{U_\alpha : \alpha \in A\}$.

Exercise 11.1.4
Show that if K is compact and $C \subset K$ is closed, then C is compact.

Proposition If $K \subset \mathbb{R}$ is compact, then K is closed.

Proof Suppose x is a limit point of K and $x \notin K$. For $n = 1, 2, 3, \ldots$, let

$$U_n = \left(-\infty, x - \frac{1}{n}\right) \cup \left(x + \frac{1}{n}, +\infty\right).$$

Then

$$\bigcup_{n=1}^{\infty} U_n = (-\infty, x) \cup (x, +\infty) \supset K.$$

However, for any $N \in \mathbb{Z}^+$, there exists $a \in K$ with

$$a \in \left(x - \frac{1}{N}, x + \frac{1}{N}\right),$$

and hence

$$a \notin \bigcup_{n=1}^{N} U_n = \left(-\infty, x - \frac{1}{N}\right) \cup \left(x + \frac{1}{N}, +\infty\right).$$

Thus the open cover $\{U_n : n \in \mathbb{Z}^+\}$ does not have a finite subcover, contradicting the assumption that K is compact.

Proposition If $K \subset \mathbb{R}$ is compact, then K is bounded.

Proof Suppose K is not bounded. For $n = 1, 2, 3, \ldots$, let $U_n = (-n, n)$. Then

$$\bigcup_{n=1}^{\infty} U_n = (-\infty, \infty) \supset K.$$

But, for any integer N, there exists $a \in K$ such that $|a| > N$, from which it follows that

$$a \notin \bigcup_{n=1}^{N} U_n = (-N, N).$$

Thus the open cover $\{U_n : n \in \mathbb{Z}^+\}$ does not have a finite subcover, contradicting the assumption that K is compact.

Taken together, the previous three propositions yield the *Heine-Borel Theorem.*

Theorem (Heine-Borel Theorem) A set $K \subset \mathbb{R}$ is compact if and only if K is closed and bounded.

11.2 Further characterizations of compactness

Proposition If $K \subset \mathbb{R}$ is compact and $\{x_n\}_{n \in I}$ is a sequence with $x_n \in K$ for every $n \in I$, then $\{x_n\}_{n \in I}$ has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with

$$\lim_{k \to \infty} x_{n_k} \in K.$$

Proof Since K is bounded, $\{x_n\}_{n \in I}$ has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$. Since K is closed, we must have

$$\lim_{k \to \infty} x_{n_k} \in K.$$

Proposition Suppose $K \subset \mathbb{R}$ is such that whenever $\{x_n\}_{n \in I}$ is a sequence with $x_n \in K$ for every $n \in I$, then $\{x_n\}_{n \in I}$ has a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with

$$\lim_{k \to \infty} x_{n_k} \in K.$$

Then K is compact.

Proof Suppose K is unbounded. Then we may construct a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in K$ and $|x_n| > n$ for $n = 1, 2, 3, \ldots$. Hence the only possible subsequential limits of $\{x_n\}_{n=1}^{\infty}$ would be $-\infty$ and $+\infty$, contradicting our assumptions. Thus K must be bounded.

Now suppose $\{x_n\}_{n \in I}$ is a convergent sequence with $x_n \in K$ for all $n \in I$. If $L = \lim_{n \to \infty} x_n$, then L is the only subsequential limit of $\{x_n\}_{n \in I}$. Hence, by the assumptions of the proposition, $L \in K$. Hence K is closed.

Since K is both closed and bounded, it is compact.

Exercise 11.2.1
Show that a set $K \subset \mathbb{R}$ is compact if and only if every infinite subset of K has a limit point in K.

Exercise 11.2.2
Show that if K is compact, then $\sup K \in K$ and $\inf K \in K$.

Theorem Given a set $K \subset \mathbb{R}$, the following are equivalent:

(a) Every open cover of K has a finite subcover.
(b) Every sequence in K has a subsequential limit in K.
(c) Every infinite subset of K has a limit point in K.

Exercise 11.2.3
Suppose $K_1, K_2, K_3, \ldots$ are nonempty compact sets with $K_{n+1} \subset K_n$ for $n = 1, 2, 3, \ldots$. Show that $\bigcap_{n=1}^{\infty} K_n$ is nonempty.

Exercise 11.2.4
We say a collection of sets $\{D_\alpha : \alpha \in A\}$ has the *finite intersection property* if for every finite set $B \subset A$, $\bigcap_{\alpha \in B} D_\alpha \neq \emptyset$. Show that a set $K \subset \mathbb{R}$ is compact if and only for any collection $\{C_\alpha : \alpha \in A\}$ of closed sets in K which has the finite intersection property we have $\bigcap_{\alpha \in A} C_\alpha \neq \emptyset$.