Lecture 10: Topology of the Real Line

10.1 Intervals

Definition Given any two extended real numbers $a < b$, the set

$$(a, b) = \{x : x \in \mathbb{R}, a < x < b\}$$

is called an *open interval*. Given any two finite real numbers $a \leq b$, the sets

$$[a, b] = \{x : x \in \mathbb{R}, a \leq x \leq b\},$$

$$(-\infty, b] = \{x : x \in \mathbb{R}, x \leq b\},$$

and

$$[a, +\infty) = \{x : x \in \mathbb{R}, x \geq a\}$$

are called *closed intervals*. Any set which is an open interval, a closed interval, or is given by, for some finite real numbers $a < b$,

$$(a, b] = \{x : x \in \mathbb{R}, a < x \leq b\}$$

or

$$[a, b) = \{x : x \in \mathbb{R}, a \leq x < b\},$$

is called an *interval*.

Proposition If $a, b \in \mathbb{R}$, then

$$(a, b) = \{x : x = \lambda a + (1 - \lambda)b, 0 < \lambda < 1\}.$$

Proof Suppose $x = \lambda a + (1 - \lambda)b$ for some $0 < \lambda < 1$. Then

$$b - x = \lambda b - \lambda a = \lambda(b - a) > 0,$$

so $x < b$, and

$$x - a = (\lambda - 1)a - (1 - \lambda)b = (1 - \lambda)(b - a) > 0,$$

so $a < x$. Hence $x \in (a, b)$.

Now suppose $x \in (a, b)$. Then

$$x = \left(\frac{b - x}{b - a}\right) a + \left(\frac{x - a}{b - a}\right) b = \left(\frac{b - x}{b - a}\right) a + \left(1 - \frac{b - x}{b - a}\right) b$$

and

$$0 < \frac{b - x}{b - a} < 1.$$

10.2 Open sets

Definition A set $U \subset \mathbb{R}$ is said to be *open* if for every $x \in U$ there exists $\epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \subset U.$$

Proposition Every open interval I is an open set.

Proof Suppose $I = (a, b)$ where $a < b$ are extended real numbers. Given $x \in I$, let ϵ be the smaller of $x - a$ and $b - x$. Suppose $y \in (x - \epsilon, x + \epsilon)$. If $b = +\infty$, then $b > y$; otherwise, we have

$$b - y > b - (x + \epsilon) = (b - x) - \epsilon \geq (b - x) - (b - x) = 0,$$

so $b > y$. If $a = -\infty$, then $a < y$; otherwise,

$$y - a > (x - \epsilon) - a = (x - a) - \epsilon \geq (x - a) - (x - a) = 0,$$

so $a < y$. Thus $y \in I$ and I is an open set.

Note that $\mathbb{R}$ is an open set (it is, in fact, the open interval $(-\infty, +\infty)$), as is $\emptyset$ (it satisfies the definition trivially).

Proposition Suppose A is a set and, for each $\alpha \in A$, U_α is an open set. Then $\bigcup_{\alpha \in A} U_\alpha$ is an open set.

Proof Let $x \in \bigcup_{\alpha \in A} U_\alpha$. Then $x \in U_\alpha$ for some $\alpha \in A$. Since U_α is open, there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U_\alpha$. Thus

$$(x - \epsilon, x + \epsilon) \subset U_\alpha \subset \bigcup_{\alpha \in A} U_\alpha.$$

Hence $\bigcup_{\alpha \in A} U_\alpha$ is open.

Proposition Suppose $U_1, U_2, \ldots, U_n$ is a finite collection of open sets. Then $\bigcap_{i=1}^n U_i$ is open.

Proof Let $x \in \bigcap_{i=1}^n U_i$. Then $x \in U_i$ for every $i = 1, 2, \ldots, n$. For each i, choose $\epsilon_i > 0$ such that $(x - \epsilon_i, x + \epsilon_i) \subset U_i$. Let ϵ be the smallest of $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$. Then $\epsilon > 0$ and

$$(x - \epsilon, x + \epsilon) \subset (x - \epsilon_i, x + \epsilon_i) \subset U_i$$

for every $i = 1, 2, \ldots, n$. Thus

$$(x - \epsilon, x + \epsilon) \subset \bigcap_{i=1}^n U_i.$$

Hence $\bigcap_{i=1}^n U_i$ is an open set.

Along with the facts that $\mathbb{R}$ and $\emptyset$ are both open sets, the last two propositions show that the collection of open subsets of $\mathbb{R}$ form a *topology.*

Definition Let $A \subset \mathbb{R}$. We say $x \in A$ is an *interior* point of A if there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset A$. The set of all interior points of A is called the *interior* of A, denoted A°.

Exercise 10.2.1
Show that if $A \subset \mathbb{R}$, then A° is open.

Exercise 10.2.2
Show that A is open if and only if $A = A^\circ$.

Exercise 10.2.3
Let $U \subset \mathbb{R}$ be a nonempty open set. Show that $\sup U \notin U$ and $\inf U \notin U$.

10.3 Closed sets

Definition A point $x \in \mathbb{R}$ is called a *limit point* of a set $A \subset \mathbb{R}$ if for every $\epsilon > 0$ there exists $a \in A$, $a \neq x$, such that $a \in (x - \epsilon, x + \epsilon)$.

Definition Suppose $A \subset \mathbb{R}$. A point $a \in A$ is called an *isolated point* of A if there exists an $\epsilon > 0$ such that

$$A \cap (x - \epsilon, x + \epsilon) = \{a\}.$$

Exercise 10.3.1
Identify the limit points and isolated points of the following sets:
(a) $[-1, 1]$,
(b) $(-1, 1)$,
(c) $\left\{\frac{1}{n} : n \in \mathbb{Z}^+\right\}$,
(d) $\mathbb{Z}$,
(e) $\mathbb{Q}$.

Exercise 10.3.2
Suppose x is a limit point of the set A. Show that for every $\epsilon > 0$, the set $(x - \epsilon, x + \epsilon) \cap A$ is infinite.

We denote the set of limit points of a set A by A'.

Definition Given a set $A \subset \mathbb{R}$, the set $\overline{A} = A \cup A'$ is called the *closure* of A.

Definition A set $C \subset \mathbb{R}$ is said to be *closed* if $C = \overline{C}$.

Proposition If $A \subset \mathbb{R}$, then $\overline{A}$ is closed.

Proof Suppose x is a limit point of $\overline{A}$. We will show that x is a limit point of A, and hence $x \in \overline{A}$. Now for any $\epsilon > 0$, there exists $a \in \overline{A}$, $a \neq x$, such that

$$a \in \left(x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}\right).$$

If $a \notin A$, then a is a limit point of A, so there exists $b \in A$, $b \neq a$ and $b \neq x$, such that

$$b \in \left(a - \frac{\epsilon}{2}, a + \frac{\epsilon}{2}\right).$$

Then

$$|x - b| \leq |x - a| + |a - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $x \in A'$, and so $\overline{A}$ is closed.

Proposition A set $C \subset \mathbb{R}$ is closed if and only if for every convergent sequence $\{a_k\}_{k \in K}$ with $a_k \in C$ for all $k \in K$,

$$\lim_{k \to \infty} a_k \in C.$$

Proof Suppose C is closed and $\{a_k\}_{k \in K}$ is a convergent sequence with $a_k \in C$ for all $k \in K$. Let $x = \lim_{k \to \infty} a_k$. If $x = a_k$ for some integer k, then $x \in C$. Otherwise, for every $\epsilon > 0$, there exists an integer N such that $|a_N - x| < \epsilon$. Hence $a_N \neq x$ and

$$a_N \in (x - \epsilon, x + \epsilon).$$

Thus x is a limit point of C, and so $x \in C$ since C is closed.

Now suppose that for every convergent sequence $\{a_k\}_{k \in K}$ with $a_k \in C$ for all $k \in K$, $\lim_{k \to \infty} a_k \in C$. Let x be a limit point of C. For $k = 1, 2, 3, \ldots$, choose $a_k \in C$ such that $a_k \in \left(x - \frac{1}{k}, x + \frac{1}{k}\right)$. Then clearly

$$x = \lim_{k \to \infty} a_k,$$

so $x \in C$. Thus C is closed.

Exercise 10.3.3
Show that every closed interval I is a closed set.

Proposition Suppose A is a set and, for each $\alpha \in A$, C_α is a closed set. Then $\bigcap_{\alpha \in A} C_\alpha$ is a closed set.

Proof Suppose x is a limit point of $\bigcap_{\alpha \in A} C_\alpha$. Then for any $\epsilon > 0$, there exists $y \in \bigcap_{\alpha \in A} C_\alpha$ such that $y \neq x$ and $y \in (x - \epsilon, x + \epsilon)$. But then for any $\alpha \in A$, $y \in C_\alpha$, so x is a limit point of C_α. Since C_α is closed, it follows that $x \in C_\alpha$ for every $\alpha \in A$. Thus $x \in \bigcap_{\alpha \in A} C_\alpha$ and $\bigcap_{\alpha \in A} C_\alpha$ is closed.

Proposition Suppose $C_1, C_2, \ldots, C_n$ is a finite collection of closed sets. Then $\bigcup_{i=1}^{n} C_i$ is closed.

Proof Suppose $\{a_k\}_{k \in K}$ is a convergent sequence with $a_k \in \bigcup_{i=1}^{n} C_i$ for every $k \in K$. Let $L = \lim_{k \to \infty} a_k$. Since K is an infinite set, there must exist an integer m and a subsequence $\{a_{n_j}\}_{j=1}^{\infty}$ such that $a_{n_j} \in C_m$ for $j = 1, 2, \ldots$. Since every subsequence of $\{a_k\}_{k \in K}$ converges to L, $\{a_{n_j}\}_{j=1}^{\infty}$ must converge to L. Since C_m is closed,

$$L = \lim_{j \to \infty} a_{n_j} \in C_m \subset \bigcup_{i=1}^{n} C_i.$$

Thus $\bigcup_{i=1}^n C_i$ is closed.

Note that both $\mathbb{R}$ and $\emptyset$ satisfy the definition of a closed set.

Proposition A set $C \subset \mathbb{R}$ is closed if and only if $\mathbb{R} \setminus C$ is open.

Proof Assume C is closed and let $U = \mathbb{R} \setminus C$. If $C = \mathbb{R}$, then $U = \emptyset$, which is open; if $C = \emptyset$, then $U = \mathbb{R}$, which is open. So we may assume both C and U are nonempty. Let $x \in U$. Then x is not a limit point of C, so there exists an $\epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \cap C = \emptyset.$$

Thus

$$(x - \epsilon, x + \epsilon) \subset U,$$

so U is open.

Now suppose $U = \mathbb{R} \setminus C$ is open. If $U = \mathbb{R}$, then $C = \emptyset$, which is closed; if $U = \emptyset$, then $C = \mathbb{R}$, which is closed. So we may assume both U and C are nonempty. Let x be a limit point of C. Then, for every $\epsilon > 0$,

$$(x - \epsilon, x + \epsilon) \cap C \neq \emptyset.$$

Hence there does not exist $\epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \subset U.$$

Thus $x \notin U$, so $x \in C$ and C is closed.

Exercise 10.3.4
For $n = 1, 2, 3, \ldots$, let $I_n = (-\frac{1}{n}, \frac{n+1}{n})$. Is $\bigcap_{n=1}^\infty I_n$ open or closed?

Exercise 10.3.5
For $n = 3, 4, 5, \ldots$, let $I_n = [\frac{1}{n}, \frac{n-1}{n}]$. Is $\bigcup_{n=3}^\infty I_n$ open or closed?

Exercise 10.3.6
Suppose, for $n = 1, 2, 3, \ldots$, the intervals $I_n = [a_n, b_n]$ are such that $I_{n+1} \subset I_n$. If $a = \sup\{a_n : n \in \mathbb{Z}^+\}$ and $b = \inf\{b_n : n \in \mathbb{Z}^+\}$, show that

$$\bigcap_{n=1}^\infty I_n = [a, b].$$

Exercise 10.3.7
Find a sequence I_n, $n = 1, 2, 3, \ldots$, of closed intervals such that $I_{n+1} \subset I_n$ for $n = 1, 2, 3, \ldots$ and $\bigcap_{n=1}^\infty I_n = \emptyset$.

Exercise 10.3.8
Find a sequence I_n, $n = 1, 2, 3, \ldots$, of bounded, open intervals such that $I_{n+1} \subset I_n$ for $n = 1, 2, 3, \ldots$ and $\bigcap_{n=1}^\infty I_n = \emptyset$.

Exercise 10.3.9
Suppose $A_i \subset \mathbb{R}$, $i = 1, 2, \ldots, n$, and let $B = \bigcup_{i=1}^{n} A_i$. Show that $\overline{B} = \bigcup_{i=1}^{n} \overline{A_i}$.

Exercise 10.3.10
Suppose $A_i \subset \mathbb{R}$, $i \in \mathbb{Z}^+$, and let $B = \bigcup_{i=1}^{\infty} A_i$. Show that $\bigcup_{i=1}^{\infty} \overline{A_i} \subset \overline{B}$. Find an example for which $\overline{B} \neq \bigcup_{i=1}^{\infty} \overline{A_i}$.

Exercise 10.3.11
Suppose $U \subset \mathbb{R}$ is a nonempty open set.For each $x \in U$, let

$$J_x = \bigcup (x - \epsilon, x + \delta),$$

where the union is taken over all $\epsilon > 0$ and $\delta > 0$ such that $(x - \epsilon, x + \delta) \subset U$.
(a) Show that for every $x, y \in U$, either $J_x \cap J_y = \emptyset$, or $J_x = J_y$.
(b) Show that $U = \bigcup_{x \in B} J_x$, where $B \subset U$ is either finite or countable.