Lecture 1: Sets and Relations

1.1 The integers

Kronecker once said, "God made the integers; all the rest is the work of man." Taking this as our starting point, we assume the existence of the set

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\},$$

the set of integers. Moreover, we assume the properties of the operations of addition and multiplication of integers, along with other elementary properties such as the Fundamental Theorem of Arithmetic (i.e., every integer may be factored into a product of prime numbers and this factorization is essentially unique).

1.2 Sets

We will take a naive view of sets: given any property p, we may determine a set by collecting together all objects which have property p. This may be done either by explicit enumeration, such as, p is the property of being one of a, b, or c, which creates the set $\{a, b, c\}$, or by stating the desired property, such as, p is the property of being a positive integer, which creates the set

$$\mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\}.$$

The notation $x \in A$ indicates that x is an element of the set A. Given sets A and B, we say A is a *subset* of B, denoted $A \subset B$, if from the fact that $x \in A$ it necessarily follows that $x \in B$. We say sets A and B are *equal* if both $A \subset B$ and $B \subset A$.

Given two sets A and B, the set

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

is called the *union* of A and B and the set

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

is called the *intersection* of A and B. The set

$$A \setminus B = \{x : x \in A, x \notin B\}$$

is called the *difference* of A and B.

More generally, if I is a set and $\{A_\alpha : \alpha \in I\}$ is a collection of sets, one for each element of I, then we have the union

$$\bigcup_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for some } \alpha\}$$

and the intersection

$$\bigcap_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for all } \alpha\}.$$

For example, if $I = \{2, 3, 4, \ldots\}$ and, for each $i \in I$,

$$A_i = \{n : n \in \mathbb{Z}, n > i, n \text{ is not divisible by } i\},$$

then $\bigcap_{i \in I} A_i$ is the set of prime numbers.

If A and B are both sets, the set

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

is called the *cartesian product* of A and B. If $A = B$, we write

$$A^2 = A \times A.$$

For example,

$$\mathbb{Z}^2 = \{(m, n) : m \in \mathbb{Z}, n \in \mathbb{Z}\}$$

is the set of all ordered pairs of integers.

Given two sets A and B, a subset R of $A \times B$ is called a *relation*. Given a relation R, we will write $a \sim_R b$, or simply $a \sim b$ if R is clear from the context, to indicate that $(a, b) \in R$. For example, we could define a relation $R \subset \mathbb{Z}^2$ by specifying that $(m, n) \in R$, that is, $m \sim_R n$, if m divides n.

Consider a set A and a relation $R \subset A^2$. For purposes of conciseness, we say simply that R is a relation *on* A. If R is such that $a \sim_R a$ for every $a \in A$, we say R is *reflexive*; if R is such that $b \sim_R a$ whenever $a \sim_R b$, we say R is *symmetric*; if $a \sim_R b$ and $b \sim_R c$ together imply $a \sim_R c$, we say R is *transitive*. A relation which is reflexive, symmetric, and transitive is called an *equivalence relation*.

Exercise 1.2.1
Show that the relation R on $\mathbb{Z}$ defined by $m \sim_R n$ if m divides n is reflexive and transitive, but not symmetric.

Exercise 1.2.2
Show that the relation R on $\mathbb{Z}$ defined by $m \sim_R n$ if $m - n$ is even is an equivalence relation.

Given an equivalence relation R on a set A and an element $x \in A$, we call

$$[x] = \{y : y \in A, y \sim_R x\}$$

the *equivalence class* of x.

Exercise 1.2.3
Given an equivalence relation R on a set A, show that

(a) $[x] \cap [y] \neq \emptyset$ if and only if $x \sim_R y$;
(b) $[x] = [y]$ if and only if $x \sim_R y$.

As a consequence of the previous exercise, the equivalence classes of a an equivalence relation on a set A constitute a *partition* of A, that is, A may be written as the disjoint union of equivalence classes.

Exercise 1.2.4
Find the equivalence classes for the equivalence relation in Exercise 1.2.2.