Lecture 9: Limits

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9.1 Limit of a function

Definition 9.1. Suppose $S \subset \mathbb{C}$, $f : S \to \mathbb{C}$, and z_0 is an accumulation point of S. We say that the *limit* of $f(z)$ as z approaches z_0 is w_0 if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon$$

whenever $z \in S$ and

$$0 < |z - z_0| < \delta.$$

We write either $\lim_{z \to z_0} f(z) = w_0$ or $f(z) \to w_0$ as $z \to z_0$.

Equivalently, the definition says that given any ϵ neighborhood V of w_0, there exists a deleted δ neighborhood U of z_0 such that $f(z) \in V$ whenever $z \in U \cap S$. The assumption that z_0 is an accumulation point of S guarantees that $U \cap S \neq \emptyset$.

Note that if S is a region, then z_0 may be any point either in S or in the boundary of S.

Proposition 9.1. Suppose $S \subset \mathbb{C}$ and $f : S \to \mathbb{C}$. If

$$\lim_{z \to z_0} f(z) = w_0$$

and

$$\lim_{z \to z_0} f(z) = w_1,$$

then $w_0 = w_1$.

Proof. Suppose $w_0 \neq w_1$ and let

$$\epsilon = \frac{|w_0 - w_1|}{2}.$$

Then $\epsilon > 0$, so there exists $\delta_1 > 0$ such that

$$|f(z) - w_0| < \epsilon$$

whenever $z \in S$ and $0 < |z - z_0| < \delta_1$ and there exists $\delta_2 > 0$ such that

$$|f(z) - w_1| < \epsilon$$

whenever $z \in S$ and $0 < |z - z_0| < \delta_2$. Let δ be the smaller of δ_1 and δ_2. Then for $z \in S$ with $0 < |z - z_0| < \delta$,

$$|w_0 - w_1| = |(f(z) - w_1) - (f(z) - w_0)| \leq |f(z) - w_1| + |f(z) - w_0| < 2\epsilon,$$

contradicting the choice of ϵ. □

Example 9.1. Suppose $c \in \mathbb{C}$ and define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = c$. We will show that, for any $z_0 \in \mathbb{C}$,

$$\lim_{z \to z_0} f(z) = c.$$

Given $\epsilon > 0$, we need to find $\delta > 0$ such that

$$|f(z) - c| < \epsilon$$

whenever

$$0 < |z - z_0| < \delta.$$

Since $|f(z) - c| = |c - c| = 0$ for all z, clearly any value of δ will work.

Example 9.2. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z$. We will show that, for any $z_0 \in \mathbb{C}$,

$$\lim_{z \to z_0} f(z) = z_0.$$

Given $\epsilon > 0$, we need to find $\delta > 0$ such that

$$|f(z) - z_0| < \epsilon$$

whenever

$$0 < |z - z_0| < \delta.$$

Since $|f(z) - z_0| = |z - z_0|$ for all z, we will obtain the desired result by setting $\delta = \epsilon$.

9.2 Properties of limits

Proposition 9.2. Suppose $f : S \to \mathbb{C}$ and $g : S \to \mathbb{C}$. If

$$\lim_{z \to z_0} f(z) = w_0 \text{ and } \lim_{z \to z_0} g(z) = w_1,$$

then

$$\lim_{z \to z_0} (f(z) + g(z)) = w_0 + w_1.$$

Proof. Given $\epsilon > 0$, there exists $\delta_1 > 0$ such that

$$|f(z) - w_0| < \frac{\epsilon}{2}$$

whenever $z \in S$ and $0 < |z - z_0| < \delta_1$ and there exists $\delta_2 > 0$ such that

$$|g(z) - w_1| < \frac{\epsilon}{2}$$

whenever $z \in S$ and $0 < |z - z_0| < \delta_2$. Let δ be the smaller of δ_1 and δ_2. Then

$$|(f(z) + g(z)) - (w_0 + w_1)| \leq |f(z) - w_0| + |g(z) - w_1| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

whenever $z \in S$ and $0 < |z - z_0| < \delta$. Hence

$$\lim_{z \to z_0} (f(z) + g(z)) = w_0 + w_1.$$

□

Proposition 9.3. Suppose $f : S \to \mathbb{C}$ and $g : S \to \mathbb{C}$. If

$$\lim_{z \to z_0} f(z) = w_0 \text{ and } \lim_{z \to z_0} g(z) = w_1,$$

then

$$\lim_{z \to z_0} (f(z)g(z)) = w_0 w_1.$$

Proof. We first note that

$$\begin{aligned} |f(z)g(z) - w_0 w_1| &= |f(z)g(z) - w_0 g(z) + w_0 g(z) - w_0 w_1| \\ &= |g(z)(f(z) - w_0) + w_0(g(z) - w_1)| \end{aligned}$$

$$\leq |g(z)||f(z) - w_0| + |w_0||g(z) - w_1|.$$

Now we may choose $\delta_1 > 0$ such that

$$|g(z) - w_1| < 1$$

whenever $z \in S$ and $0 < |z - z_0| < \delta_1$. It follows that

$$|g(z)| = |(g(z) - w_1) + w_1| \leq |g(z) - w_1| + |w_1| < 1 + |w_1|$$

whenever $z \in S$ and $0 < |z - z_0| < \delta_1$. Moreover, we may choose $\delta_2 > 0$ such that

$$|f(z) - w_0| < \frac{\epsilon}{2(1 + |w_1|)}$$

whenever $z \in S$ and $0 < |z - z_0| < \delta_2$ and we may choose $\delta_3 > 0$ such that

$$|g(z) - w_1| < \frac{\epsilon}{2(1 + |w_0|)}$$

whenever $z \in S$ and $0 < |z - z_0| < \delta_3$. Now let δ be the smaller of δ_1, δ_2, and δ_3. If $z \in S$ and $0 < |z - z_0| < \delta$, then

$$|g(z)||f(z) - w_0| < (1 + |w_1|)\frac{\epsilon}{2(1 + |w_1|)} = \frac{\epsilon}{2}$$

and

$$|w_0||g(z) - w_1| < (1 + |w_0|)\frac{\epsilon}{2(1 + |w_0|)} = \frac{\epsilon}{2}.$$

Hence

$$|f(z)g(z) - w_0 w_1| < \epsilon$$

whenever $z \in S$ and $0 < |z - z_0| < \delta$, and so

$$\lim_{z \to z_0} f(z)g(z) = w_0 w_1.$$

□

Proposition 9.4. Suppose $f : S \to \mathbb{C}$ and $g : S \to \mathbb{C}$. If

$$\lim_{z \to z_0} f(z) = w_0 \text{ and } \lim_{z \to z_0} g(z) = w_1,$$

and $w_1 \neq 0$, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{w_0}{w_1}.$$

Proof. We first note that

$$\begin{aligned}\left|\frac{f(z)}{g(z)} - \frac{w_0}{w_1}\right| &= \left|\frac{w_1 f(z) - w_0 g(z)}{w_1 g(z)}\right| \\ &= \frac{|w_1 f(z) - w_0 w_1 + w_0 w_1 - w_0 g(z)|}{|w_1||g(z)|} \\ &\leq \frac{|w_1||f(z) - w_0| + |w_0||g(z) - w_1|}{|w_1||g(z)|}.\end{aligned}$$

If we choose δ_1 so that

$$|g(z) - w_1| < \frac{|w_1|}{2}$$

whenever $z \in S$ and $0 < |z - z_0| < \delta_1$, then

$$|g(z)| = |(g(z) - w_1) + w_1| \geq ||w_1| - |g(z) - w_1|| = |w_1| - |g(z) - w_1| > \frac{|w_1|}{2}$$

whenever $z \in S$ and $0 < |z - z_0| < \delta_1$. It follows that for such values of z,

$$\left|\frac{f(z)}{g(z)} - \frac{w_0}{w_1}\right| < \frac{2}{|w_1|}|f(z) - w_0| + \frac{2|w_0|}{|w_1|^2}|g(z) - w_1|.$$

Now choose $\delta_2 > 0$ such that

$$|f(z) - w_0| < \frac{|w_1|\epsilon}{4}$$

whenever $z \in S$ and $0 < |z - z_0| < \delta_2$ and, if $|w_0| \neq 0$, $\delta_3 > 0$ such that

$$|g(z) - w_1| < \frac{|w_1|^2\epsilon}{4|w_0|}$$

whenever $0 < |z - z_0| < \delta_3$. If $|w_0| = 0$, let $\delta_3 = 1$. It now follows that if δ is the smallest of δ_1, δ_2, and δ_3, then

$$\left|\frac{f(z)}{g(z)} - \frac{w_0}{w_1}\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and so

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{w_0}{w_1}.$$

$\square$

Proposition 9.5. If P is a polynomial, then for any $z_0 \in \mathbb{C}$,

$$\lim_{z \to z_0} P(z) = P(z_0).$$

If R is a rational function and $R(z_0) \neq 0$, then

$$\lim_{z \to z_0} R(z) = R(z_0).$$

Proof. The result is an immediate consequence of the previous propositions combined with the limits

$$\lim_{z \to z_0} c = c$$

for any constant $c \in \mathbb{C}$ and

$$\lim_{z \to z_0} z = z_0.$$

□

Example 9.3. We may now compute

$$\lim_{z \to 2i} \frac{z^2+1}{z^3+4i} = \frac{(2i)^2+1}{(2i)^3+4i} = \frac{-3}{-4i} = -\frac{3}{4}i.$$

Proposition 9.6. Suppose $S \subset \mathbb{C}$, $f : S \to \mathbb{C}$, $f(x+iy) = u(x,y) + iv(x,y)$, $z_0 = x_0 + iy_0$, and $w_0 = u_0 + iv_0$. Then

$$\lim_{z \to z_0} f(z) = w_0$$

if and only if

$$\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0$$

and

$$\lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0.$$

Proof. One direction follows from our earlier results: if

$$\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0$$

and

$$\lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0,$$

then

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} u(z) + i \lim_{z\to z_0} v(z) = u_0 + iv_0 = w_0.$$

For the other direction, suppose

$$\lim_{z\to z_0} f(z) = w_0.$$

Then we may choose $\epsilon > 0$ such that

$$|f(z) - w_0| < \epsilon$$

whenever $z \in S$ and $0 < |z - z_0| < \delta$. For such z, it follows that, with $z = x + iy$,

$$|u(x, y) - u_0| \leq |f(z) - w_0| < \epsilon$$

and

$$|v(x, y) - v_0| \leq |f(z) - w_0| < \epsilon.$$

Hence

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0$$

and

$$\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0.$$

□

Example 9.4. Suppose $f(x+iy) = 4xy+i\sqrt{x + y}$. Then, using limit results from calculus,

$$\lim_{z\to 5-3i} f(z) = \lim_{(x,y)\to(5,-3)} 4xy + i \lim_{(x,y)\to(5,-3)} \sqrt{x + y} = -60 + i\sqrt{2}.$$

Example 9.5. Suppose

$$f(z) = \frac{z}{\bar{z}}.$$

Note that if $z = x$, $x \neq 0$, then

$$f(z) = \frac{x}{x} = 1,$$

whereas if $z = iy$, $y \neq 0$,

$$f(z) = \frac{iy}{-iy} = -1.$$

Hence $f(z) \to 1$ as $z \to 0$ along the real-axis, while $f(z) \to -1$ as $z \to 0$ along the imaginary axis. Hence $f(z)$ does not have a limit as z approaches 0.