Lecture 8:
Mappings

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8.1 Visualizing functions

Recall that the graph of a function $f : \mathbb{R} \to \mathbb{R}$ or $f : \mathbb{R}^2 \to \mathbb{R}$ provides a visualization of the behavior of f. However, if $S \subset \mathbb{C}$ and $f : S \to \mathbb{C}$, then the graph of f is in four-dimensional space, and so is not easily visualized. As one alternative, we may consider $w = f(z)$ as a transformation, or *mapping*, which takes a region in the z-plane and maps it to a region in the w-plane. If $z \in S$, we call $w = f(z)$ the *image* of z; for a set $T \subset S$, we call

$$\{w \in \mathbb{C} : w = f(z) \text{ for some } z \in T\}$$

the *image* of T; and we call the image of S the *range* of f.

Example 8.1. A mapping $w = z + z_0$, where z_0 is a fixed constant, is a *translation*. For example, the mapping $w = z + i$ shifts every point z one unit vertically, $x + iy$ going to $x + i(y + 1)$.

Example 8.2. The mapping $w = ze^{i\theta}$, where θ is a fixed real number, is a *rotation*. For example, the mapping $w = zi = ze^{i\frac{\pi}{2}}$ is a rotation counter-clockwise through an angle $\frac{\pi}{2}$, taking the point $re^{i\theta}$ to the point $re^{i\left(\theta+\frac{\pi}{2}\right)}$.

Example 8.3. The mapping $w = \bar{z}$ is a *reflection*, taking $x+iy$ and reflecting it about the real axis to $x - iy$.

8.2 The mapping $w = z^2$

If $z = x + iy$ and $w = z^2$, then

$$w = (x + iy)^2 = (x^2 - y^2) + 2xyi.$$

Hence $w = u + iv$ where

$$u = x^2 - y^2 \text{ and } v = 2xy.$$

Consider the hyperbola H in the xy-plane with equation $x^2 - y^2 = c$, $c > 0$. On the right-hand branch of H, $x = \sqrt{y^2 + c}$, and so

$$v = 2y\sqrt{y^2 + c}.$$

It follows that as (x, y) moves along the right-hand branch of H, with y going from $-\infty$ to ∞, (u, v) moves upward along the vertical line $u = c$, with v going from $-\infty$ to ∞. That is, the mapping $w = z^2$ maps the right-hand branch of the hyperbola $x^2 - y^2 = c$ onto the vertical line $u = c$. On the left-hand branch of H, $x = -\sqrt{y^2 + c}$, and so a point (x, y) on this branch is mapped to

$$u = c \text{ and } v = -2y\sqrt{y^2 + c}.$$

It follows that as (x, y) moves along the left-hand branch of H, with y going from ∞ to $-\infty$, (u, v) moves upward along the vertical line $u = c$, with v going from $-\infty$ to ∞. That is, the mapping $z = w^2$ maps the left-hand branch of the hyperbola $x^2 - y^2 = c$ onto the vertical line $u = c$.

Now consider the hyperbola K in the xy-plane with equation $2xy = c$, $c > 0$. On the right-hand branch of K,

$$y = \frac{c}{2x},$$

and so

$$u = x^2 - \frac{c^2}{4x^2}.$$

Note that $u \to \infty$ as $x \to \infty$ and $u \to -\infty$ as $x \downarrow 0$. It follows that as (x, y) moves along the right-hand branch of K, with x going from 0 to ∞, (u, v) moves to the right along the horizontal line $v = c$. That is, the mapping $w = z^2$ maps the right-hand branch of the hyperbola $2xy = c$ onto the horizontal line $v = c$. Similarly, if as (x, y) moves along the left-hand branch of K, with x going from 0 to $-\infty$, (u, v) moves to the right along the horizontal line $v = c$.

It now follows from the work above that, for example, $w = z^2$ maps the domain

$$\{z = x + iy \in \mathbb{C} : x > 0, y > 0, xy < 1\}$$

in the xy-plane onto the domain

$$\{w = u + iv \in \mathbb{C} : 0 < v < 2\}.$$

Since $z = iy$ is mapped to $w = -y^2$, the positive imaginary axis is mapped to the negative real axis, and since $z = x$ is mapped to $w = x^2$, the positive real axis is mapped to the positive real axis. Hence $w = z^2$ maps the closed region

$$\{z = x + iy \in \mathbb{C} : x \geq 0, y \geq 0, xy \leq 1\}$$

in the xy-plane onto the closed region

$$\{w = u + iv \in \mathbb{C} : 0 \leq v \leq 2\}.$$

We may also look at the mapping $w = z^2$ using polar coordinates. If $z = re^{i\theta}$, then $w = r^2 e^{i2\theta}$. That is, if $w = \rho e^{i\varphi}$ and $w = z^2$, then

$$\rho = r^2 \text{ and } \varphi = 2\theta + 2k\pi, \text{ where } k = 0, \pm 1, \pm 2, \ldots.$$

In particular, this means $w = z^2$ maps the first quadrant of the z-plane, that is,

$$\left\{z = re^{i\theta} : r \geq 0, 0 \leq \theta \leq \frac{\pi}{2}\right\}$$

onto the upper half plane of w-plane, that is,

$$\{w = \rho e^{i\varphi} : \rho \geq 0, 0 \leq \varphi \leq \pi\}.$$

Similarly, $w = z^2$ maps

$$\{z = re^{i\theta} : r \geq 0, 0 \leq \theta \leq \pi\}$$

onto the entire w-plane.

8.3 The exponential mapping

Although we will not formally study the exponential function of a complex variable until later, we should expect that if $z = x + iy$, then

$$w = e^z = e^{x+iy} = e^x e^{iy}.$$

Since we have already defined

$$e^{iy} = \cos(y) + i\sin(y),$$

we have

$$e^z = e^x(\cos(y) + i\sin(y)).$$

Now if $w = \rho e^{i\varphi}$, then $w = e^z = e^x e^{iy}$ implies that

$$\rho = e^x \text{ and } \varphi = y + 2k\pi, \text{ where } k = 0, \pm 1, \pm 2, \ldots.$$

In particular, if z lies on the vertical line $x = c$, that is, $z = c + iy$, then w traverses the circle of radius e^c with center at the origin as y passes through every interval of length 2π. That is, $w = e^z$ maps vertical lines onto circles centered at the origin, with the mapping repeating a countercockwise traversal of the circle an infinite number of times as y goes from $-\infty$ to ∞.

If z lies on the horizontal line $y = c$, that is $z = x + ic$, then w lies on the ray $\varphi = c$. In fact, w traverses this entire ray as x goes from $-\infty$ to ∞.

As a consequence of the above observations, $w = e^z$ maps a rectangle $R = [a, b] \times [c, d]$ in the z-plane onto a circular sector

$$\{w = \rho e^{i\varphi} : e^a \leq \rho \leq e^b, c \leq \varphi \leq d\}$$

in the w-plane. For the infinite strip

$$S = \{z = x + iy : 0 \leq y \leq \pi\},$$

$w = e^z$ maps S onto

$$\{w = u + iv : v \geq 0, w \neq 0\}.$$