Lecture 5:
Roots of Complex Numbers

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5.1 Roots

Suppose z_0 is a complex number and, for some positive integer n, z is an nth root of z_0; that is, $z^n = z_0$. Now if $z = re^{i\theta}$ and $z_0 = r_0 e^{i\theta_0}$, then we must have

$$r^n = r_0 \text{ and } n\theta = \theta_0 + 2k\pi$$

for some integer k. Hence we must have

$$r = \sqrt[n]{r_0},$$

where $\sqrt[n]{r_0}$ denotes the unique positive nth root of the real number r, and

$$\theta = \frac{\theta_0}{n} + \frac{2k\pi}{n}$$

for some integer k. Hence the numbers

$$z = \sqrt[n]{r_0} \exp\left(i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right), \text{ for } k = 0, \pm 1, \pm 2, \ldots,$$

are all nth roots of z_0, with

$$z = \sqrt[n]{r_0} \exp\left(i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right), \text{ for } k = 0, 1, 2, \ldots, n-1,$$

being the n distinct nth roots of z_0. We let $z_0^{\frac{1}{n}}$ denote this set of nth roots of z_0. Note that these roots are equally spaced around the circle of radius $\sqrt[n]{r}_0$. If $\theta_0 = \operatorname{Arg} z_0$, we call

$$z = \sqrt[n]{r}_0 e^{i\frac{\theta_0}{n}},$$

the *principal root* of z_0.

5.2 Roots of unity

Example 5.1. Since

$$1 = 1e^{i(0+2k\pi)} \text{ for } k = 0, \pm 1, \pm 2, \ldots,$$

for any positive integer n,

$$z = e^{i\frac{2k\pi}{n}}$$

is an nth root of 1 for any integer k, which we call an nth *root of unity.* If we let

$$\omega_n = e^{i\frac{2\pi}{n}},$$

then, for any integer k,

$$\omega_n^k = e^{i\frac{2k\pi}{n}}$$

Hence the distinct nth roots of unity are

$$1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}.$$

For example, when $n = 2$,

$$\omega_2 = e^{i\pi} = -1,$$

and the roots of unity are simply 1 and -1. When $n = 3$,

$$w_3 = e^{i\frac{2\pi}{3}},$$

and the distinct third roots of unity are

$$1, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}};$$

that is

$$1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

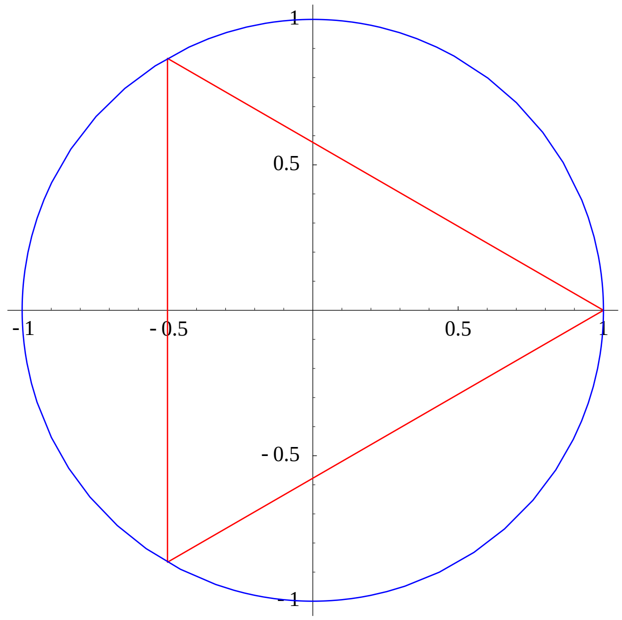

Figure 5.1: The third roots of unity form an equilateral triangle

When $n = 4$,

$$\omega_4 = e^{i\frac{2\pi}{4}} = e^{i\frac{\pi}{2}} = i,$$

and the distinct roots fourth roots of unity are

$$1, i, -1, -i.$$

Note that the nth roots of unity are equally spaced around the unit circle. For example, the third roots of unity form an equilateral triangle in the unit circle, as shown in Figure 5.1.

More generally, if c is any particular nth root of z_0, then the distinct nth roots of z_0 are

$$c, c\omega_n, c\omega_n^2, \ldots, c\omega_n^{n-1}.$$

Example 5.2. To find the cube roots of $-27i$, we first note that

$$-27i = 27\exp\left(i\left(-\frac{\pi}{2} + 2k\pi\right)\right), \text{ for } k = 0, \pm 1, \pm 2, \ldots.$$

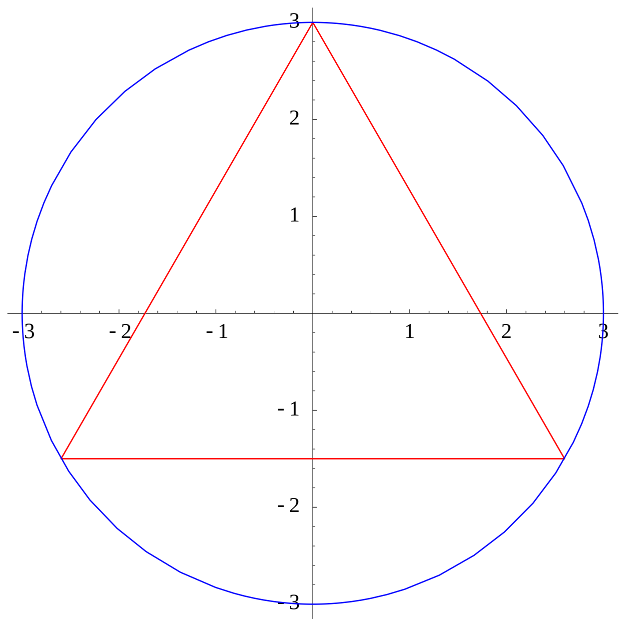

Figure 5.2: The cube roots of $-27i$ form an equilateral triangle

Hence the principal cube root of $-27i$ is

$$c_0 = 3e^{-i\frac{\pi}{6}} = 3\left(\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)\right) = \frac{3\sqrt{3}}{2} - \frac{3}{2}i.$$

The remaining cube roots are

$$c_1 = 3e^{i\frac{\pi}{2}} = 3\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 3i,$$

and

$$c_2 = 3e^{i\frac{7\pi}{6}} = 3\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right) = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i,$$

which are simply $c_0\omega_3$ and $c_0\omega_3^2$, where

$$\omega_3 = e^{i\frac{2\pi}{3}}.$$