Lecture 46: Poles

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46.1 Types of singular points

If z_0 is an isolated singular point of f, then, for some $R > 0$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

for all z with $0 < |z - z_0| < R$. We call

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

the *principal part* of f at z_0. If for some positive integer m, $b_m \neq 0$ and $b_{m+1} = b_{m+2} = \cdots = 0$, that is,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{m} \frac{b_n}{(z - z_0)^n}$$

with $b_m \neq 0$, then we say z_0 is a *pole of order* m. If $m = 1$, we say z_0 is a *simple pole*. If an infinite number of the coefficients b_n are nonzero, we say z_0 is an *essential singular point* of f. If $b_n = 0$ for all n, we say z_0 is a *removable singular point*.

Example 46.1. Since

$$\frac{\sin(z)}{z^3} = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \cdots ,$$

$f(z) = \frac{\sin(z)}{z^3}$ has a pole of order $m = 2$ at $z = 0$.

Example 46.2. Since

$$\frac{\sin(z)}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots ,$$

$f(z) = \frac{\sin(z)}{z}$ has a removable singular point at $z = 0$.

Example 46.3. Since

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n! z^n},$$

$f(z) = e^{\frac{1}{z}}$ has an essential singular point at $z = 0$.

46.2 Residues at poles

Proposition 46.1. An isolated singular point of a function f is a pole of order m if and only if there is a function φ such that φ is analytic at z_0, $\varphi(z_0) \neq 0$, and

$$f(z) = \frac{\varphi(z)}{(z - z_0)^m}.$$

Moreover, in this case

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\varphi^{m-1}(z_0)}{(m-1)!}.$$

Example 46.4. If

$$f(z) = \frac{1}{(z+1)(z-1)^3},$$

then we may write

$$f(z) = \frac{\varphi(z)}{(z-1)^3}$$

where

$$\varphi(z) = \frac{1}{z+1}.$$

Hence $z = 1$ is a pole of order $m = 3$ and

$$\operatorname*{Res}_{z=1} f(z) = \frac{\varphi''(1)}{2} = \frac{1}{8},$$

as we have seen before. For the residue at $z = -1$, we write

$$f(z) = \frac{\varphi(z)}{z+1}$$

where

$$\varphi(z) = \frac{1}{(z-1)^3}.$$

Hence $z = -1$ is a simple pole and we have

$$\operatorname*{Res}_{z=-1} f(z) = \varphi(-1) = -\frac{1}{8}.$$

Example 46.5. Let

$$f(z) = \frac{1}{z^4+1} = \frac{1}{\left(z - \frac{1+i}{\sqrt{2}}\right)\left(z - \frac{-1+i}{\sqrt{2}}\right)\left(z - \frac{-1-i}{\sqrt{2}}\right)\left(z - \frac{1-i}{\sqrt{2}}\right)}.$$

Then

$$\operatorname*{Res}_{z=\frac{1+i}{\sqrt{2}}} f(z) = \frac{1}{\left(\frac{1+i}{\sqrt{2}} - \frac{-1+i}{\sqrt{2}}\right)\left(\frac{1+i}{\sqrt{2}} - \frac{-1-i}{\sqrt{2}}\right)\left(\frac{1+i}{\sqrt{2}} - \frac{1-i}{\sqrt{2}}\right)} = \frac{-1-i}{4\sqrt{2}}$$

and

$$\operatorname*{Res}_{z=\frac{-1+i}{\sqrt{2}}} f(z) = \frac{1}{\left(\frac{-1+i}{\sqrt{2}} - \frac{1+i}{\sqrt{2}}\right)\left(\frac{-1+i}{\sqrt{2}} - \frac{-1-i}{\sqrt{2}}\right)\left(\frac{-1+i}{\sqrt{2}} - \frac{1-i}{\sqrt{2}}\right)} = \frac{1-i}{4\sqrt{2}}$$

Hence if $R > 1$ and C is the contour, with positive orientation, consisting of the upper half of the circle $|z| = R$ and the segment along the real axis from $-R$ to R, then

$$\int_C \frac{1}{z^4+1} dz = 2\pi i \left(-\frac{1}{2\sqrt{2}} i\right) = \frac{\pi}{\sqrt{2}}.$$

Now if C_R is the upper half of the circle $|z| = R$, then

$$\int_{C_R} \frac{1}{z^4+1} dz \leq \frac{1}{R^4-1} \cdot 2\pi R = \frac{2\pi R}{R^4+1}.$$

Hence

$$\lim_{R\to\infty} \int_{C_R} \frac{1}{z^4+1} dz = 0.$$

But

$$\frac{\pi}{\sqrt{2}} = \int_C \frac{1}{z^4+1}dz = \int_{-R}^{R} \frac{1}{x^4+1}dx + \int_{C_R} \frac{1}{z^4+1}dz,$$

and so

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^4+1}dx = \frac{\pi}{\sqrt{2}} - \lim_{R\to\infty} \int_{C_R} \frac{1}{z^4+1}dz = \frac{\pi}{\sqrt{2}}.$$

It follows that

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}dx = \frac{\pi}{\sqrt{2}}.$$