Lecture 45: Cauchy's Residue Theorem

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45.1 Cauchy's residue theorem

The following result, *Cauchy's residue theorem*, follows from our previous work on integrals.

Theorem 45.1. Suppose C is a positively oriented, simple closed contour. If f is analytic on and inside C except for the finite number of singular points $z_1, z_2, \ldots, z_n$, then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

Example 45.1. Suppose C is the circle $|z| = 2$ with positive orientation and we wish to evaluate

$$\int_C \frac{1}{(z+1)(z-1)^3} dz.$$

In a previous example we saw that

$$\frac{1}{(z+1)(z-1)^3} = -\frac{1}{8(z+1)} + \frac{1}{8(z-1)} - \frac{1}{4(z-1)^2} + \frac{1}{2(z-1)^3},$$

from which it follows that

$$\operatorname*{Res}_{z=-1} \frac{1}{(z+1)(z-1)^3} = -\frac{1}{8}$$

$$\operatorname*{Res}_{z=1} \frac{1}{(z+1)(z-1)^3} = \frac{1}{8}.$$

Hence

$$\int_C \frac{1}{(z+1)(z-z)^3} dz = 2\pi i \left(-\frac{1}{8} + \frac{1}{8} \right) = 0.$$

Example 45.2. Suppose C is the circle $|z| = 2$ with positive orientation and we wish to evaluate

$$\int_C \frac{3z+5}{z^2+1} dz.$$

Using partial fractions,

$$\frac{3z+5}{z^2+1} = \frac{A}{z+i} + \frac{B}{z-i} = \frac{A(z-i)+B(z+i)}{(z+i)(z-i)}$$

for some constants A and B. Hence

$$3z + 5 = A(z-i) + B(z+i).$$

When $z = -i$, we have

$$5 - 3i = -2iA$$

and when $z = i$ we have

$$5 + 3i = 2iB.$$

Hence

$$A = \frac{3}{2} + \frac{5}{2}i$$

and

$$B = \frac{3}{2} - \frac{5}{2}i.$$

Thus

$$\frac{3z+5}{z^2+1} = \frac{\frac{3}{2}+\frac{5}{2}i}{z+i} + \frac{\frac{3}{2}-\frac{5}{2}i}{z-i},$$

and so

$$\operatorname*{Res}_{z=-i} \frac{3z+5}{z^2+1} = \frac{3}{2} + \frac{5}{2}i$$

and

$$\operatorname*{Res}_{z=i} \frac{3z+5}{z^2+1} = \frac{3}{2} - \frac{5}{2}i.$$

Hence

$$\int_C \frac{3z+5}{z^2+1} dz = 2\pi i \left(\left(\frac{3}{2} + \frac{5}{2}i \right) + \left(\frac{3}{2} - \frac{5}{2}i \right) \right) = 6\pi i.$$

45.2 Using a single residue

Theorem 45.2. Suppose f is analytic everywhere in the plane except at a finite number of singular points. If C is a positively oriented, simple closed contour containing all the singular points of f, then

$$\int_C f(z)dz = 2\pi i \operatorname*{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right).$$

Proof. Let $R_1 > 0$ be such that the circle $|z| = R_1$ contains C and let C_0 be the circle $|z| = R_0$ where $R_0 > R_1$. We first note that

$$\int_{C_0} f(z)dz = \int_C f(z)dz.$$

Moreover, we know that $f(z)$ has a Laurent series representation on the domain $|z| > R_1$; that is, for $|z| > R_1$,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

where

$$c_n = \frac{1}{2\pi i}\int_{C_0} \frac{f(z)}{z^{n+1}} dz.$$

In particular,

$$2\pi i c_{-1} = \int_{C_0} f(z)dz.$$

It follows that

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n+2}}$$

for

$$\frac{1}{|z|} > R_1,$$

that is

$$0 < |z| < \frac{1}{R_1}.$$

Hence

$$c_{-1} = \operatorname*{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right).$$

□

Example 45.3. If
$$f(z) = \frac{1}{(z+1)(z-1)^3},$$
then
$$f\left(\frac{1}{z}\right) = \frac{1}{\left(\frac{1}{z}+1\right)\left(\frac{1}{z}-1\right)^3} = -\frac{z^4}{(z+1)(z-1)^3},$$
and so
$$\frac{1}{z^2}f\left(\frac{1}{z}\right) = -\frac{z^2}{(z+1)(z-1)^3}.$$
Since this function is analytic at $z = 0$, we have
$$\operatorname*{Res}_{z=0}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right)\right) = 0.$$
Hence if C is the circle $|z| = 2$, then
$$\int_C \frac{1}{(z+1)(z-1)^3}dz = 0.$$

Example 45.4. If
$$f(z) = \frac{3z+5}{z^2+1},$$
then
$$f\left(\frac{1}{z}\right) = \frac{\frac{3}{z}+5}{\frac{1}{z^2}+1} = \frac{3z+5z^2}{1+z^2},$$
and so
$$\frac{1}{z^2}f\left(\frac{1}{z}\right) = \frac{3+5z}{z(1+z^2)}.$$
Now
$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots$$
for $|z| < 1$, and so
$$\frac{3+5z}{1+z^2} = 3 + 5z - 3z^2 - 5z^3 + 3z^4 + 5z^5 - \cdots$$
for $|z| < 1$. Thus
$$\frac{3+5z}{z(1+z^2)} = \frac{3}{z} + \frac{5}{3} - 5z^2 + 3z^3 + 5z^4 - \cdots .$$

Hence if C is the circle $|z| = 2$, with positive orientation,

$$\int_C \frac{3z+5}{z^2+1} dz = 6\pi i.$$