Lecture 39:
Uniform Convergence

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Mathematics 39

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39.1 Uniform convergence

Definition 39.1. Let $S \subset \mathbb{C}$ and suppose $f_n : S \to \mathbb{C}$ is a sequence of functions defined on S, $n = 1, 2, 3, \ldots$. If $f : S \to \mathbb{C}$ and

$$\lim_{n\to\infty} f_n(z) = f(z)$$

for every $z \in S$, then we say f_n *converges pointwise* to f on S. If for every $\epsilon > 0$ there exists a positive integer n_0 such that

$$|f_n(z) - f(z)| < \epsilon$$

whenever $n > n_0$ for every $z \in S$, then we say f_n *converges uniformly* to f on S.

Note that if f_n converges to f pointwise, then, given a specific $z \in S$, there exists a positive integer n_0 such that

$$|f_n(z) - f(z)| < \epsilon$$

whenever $n > n_0$, but n_0 may depend on the value of z. For uniform convergence, their exists a single n_0 that works for all $z \in S$.

Example 39.1. Let $S = \left\{z \in \mathbb{C} : |z| \leq \frac{1}{2}\right\}$, let $f_n(z) = z^n$, $n = 1, 2, 3, \ldots$, and let $f(z) = 0$ for all $z \in S$. Note that

$$\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} z^n = 0$$

for every $z \in S$, and so f_n converges pointwise to f on S.

Now, given $\epsilon > 0$, choose a positive integer n_0 so that

$$\left(\frac{1}{2}\right)^{n_0} < \epsilon.$$

That is, choose n_0 larger than

$$-\frac{\ln(\epsilon)}{\ln(2)}.$$

Then, for any $z \in S$,

$$|f_n(z) - f(z)| = |z^n - 0| = |z|^n \leq \left(\frac{1}{2}\right)^n < \epsilon$$

whenever $n > n_0$. Hence f_n converges uniformly to f on S.

Example 39.2. Let $S = \{z \in \mathbb{C} : |z| < 1\}$, let $f_n(z) = z^n$, $n = 1, 2, 3, \ldots$, and let $f(z) = 0$ for all $z \in S$. As in the previous example, f_n converges pointwise to f for all $z \in S$. However, the convergence is not uniform. For suppose there were a positive integer n_0 such that, for all $z \in S$,

$$|z|^n < \frac{1}{2}$$

whenever $n > n_0$. Let $m = n_0 + 1$ and let

$$z = \sqrt[m]{\frac{1}{2}}.$$

Then $z \in S$, but

$$|z|^m = \frac{1}{2},$$

and so $|z|^n$ is not less than $\frac{1}{2}$ for all $n > n_0$.

39.2 Convergence of power series

Theorem 39.1. Suppose the power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

converges at $z = z_1$, where $z_1 \neq z_0$. If $R_1 = |z_1 - z_0|$, then the power series is absolutely convergent at each z in the open disk $D = \{z \in \mathbb{C} : |z - z_0| < R_1\}$.

Proof. Since

$$\sum_{n=0}^{\infty} a_n(z_1 - z_0)^n$$

converges, we have

$$\lim_{n\to\infty} a_n(z_1 - z_0)^n = 0.$$

In particular, there exists a positive real number M such that

$$|a_n(z_1 - z_0)^n| \leq M$$

for $n = 0, 1, 2, 3, \ldots$. Now if $z \in D$, let

$$\rho = \left| \frac{z - z_0}{z_1 - z_0} \right|$$

Then $0 \leq \rho < 1$ and, for any $n = 0, 1, 2, \ldots$,

$$|a_n(z - z_0)^n| = |a_n(z_1 - z_0)^n| \left| \frac{z - z_0}{z_1 - z_0} \right|^n \leq M\rho^n.$$

Hence the series

$$\sum_{n=0}^{\infty} |a_n(z - z_0)^n|$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} M\rho^n.$$

Thus the series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n$$

converges absolutely. □

Definition 39.2. Given a power series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

we call the largest value of $R \geq 0$ such that the series converges for all z in the disk $|z - z_0| < R$ the *radius of convergence* of the series, and we call the circle $|z - z_0| = R$ the *circle of convergence*.

Note that a power series may converge at some, all, or none of the points on the circle of convergence. Moreover, although the series must converge absolutely at all points inside the circle of convergence, convergence at points on the circle of convergence need not be absolute.

Theorem 39.2. Suppose R is the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n$$

and z_1 is a point inside the circle of convergence. If $R_1 = |z_1 - z_0|$, then the power series converges uniformly on the closed disk $D = \{z \in \mathbb{C} : |z - z_0| \leq R_1\}$.

Proof. Given $z \in D$ and a positive integer N, let

$$\rho_N(z) = \sum_{n=N}^{\infty} a_n(z - z_0)^n.$$

We need to show that, given $\epsilon > 0$, we can find a positive integer n_0, independent of the value of z, such that $|\rho_N(z)| < \epsilon$ whenever $N > n_0$. Now we know that the power series converges absolutely at z_1, and so we know we may find a positive integer n_0 such that

$$\sum_{n=N}^{\infty} |a_n(z_1 - z_0)^n| < \epsilon$$

whenever $N > n_0$. Moreover, for any positive integers $N < m$,

$$\left|\sum_{n=N}^{m} a_n(z - z_0)^n\right| \leq \sum_{n=N}^{m} |a_n||z-z_0|^n \leq \sum_{n=N}^{m} |a_n||z_1-z_0|^n = \sum_{n=N}^{m} |a_n(z_1 - z_0)^n|.$$

Hence, for $N > n_0$,

$$\begin{aligned} |\rho_N(z)| &= \left|\lim_{m\to\infty} \sum_{n=N}^{m} a_n(z - z_0)^n\right| \\ &= \lim_{m\to\infty} \left|\sum_{n=N}^{m} a_n(z - z_0)^n\right| \end{aligned}$$

$$
\begin{aligned}
&\leq \lim_{m\to\infty} \sum_{n=N}^{m} |a_n(z_1 - z_0)^n| \\
&= \sum_{n=N}^{\infty} |a_n(z_1 - z_0)^n| < \epsilon.
\end{aligned}
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$\square$