Lecture 36:
Examples of Taylor Series

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36.1 Examples of Taylor series

Example 36.1. Let $f(z) = e^z$. Then f is entire, and so its Maclaurin series will converge for all z in the plane. Now $f^{(n)}(0) = e^0 = 1$ for $n = 0, 1, 2, 3, \ldots$, and so

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots$$

for all $z \in \mathbb{C}$.

Example 36.2. It follows from the previous example that

$$e^{2z} = \sum_{n=0}^{\infty} \frac{(2z)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{n!} z^n$$

for all $z \in \mathbb{C}$. Later we will prove the uniqueness of power series representations, from which it will follow that the expression above is the Maclaurin series for e^{2z}.

Example 36.3. Similarly,

$$e^{iz} = \sum_{n=0}^{\infty} \frac{i^n}{n!} z^n$$

and

$$e^{-iz} = \sum_{n=0}^{\infty} \frac{(-1)^n i^n}{n!} z^n.$$

Hence

$$e^{iz} - e^{iz} = \sum_{n=0}^{\infty} \frac{(1-(-1)^n)i^n}{n!} z^n = \sum_{n=0}^{\infty} \frac{2i^{2n+1}}{(2n+1)!} z^{2n+1} = \sum_{n=0}^{\infty} \frac{2i(-1)^n}{(2n+1)!} z^{2n+1}.$$

Thus, for all $z \in \mathbb{C}$,

$$\sin(z) = \frac{e^{iz} + e^{-iz}}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots .$$

Example 36.4. We will see later that we may differentiate a power series as we would a polynomial, that is, term by term. From this it will follow that

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) z^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)}! = z - \frac{z^2}{2} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

for all $z \in \mathbb{C}$.

Example 36.5. We now have

$$\begin{aligned}
\sinh(z) = -i\sin(iz) = -i \sum_{n=0}^{\infty} \frac{(-1)^n i^{2n+1} z^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{2n} z^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \\
&= z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots
\end{aligned}$$

and

$$\cosh(z) = \cos(iz) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots$$

for all $z \in \mathbb{C}$.

Example 36.6. We have seen previously that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots$$

when $|z| < 1$. Hence

$$\frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n} = 1 + z^2 + z^4 + \cdots$$

and

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n} = 1 - z^2 + z^4 - z^6 + \cdots$$

for all z with $|z| < 1$.

Example 36.7. For another example using the geometric series,

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{1-(1-z)} \\
&= \sum_{n=0}^{n} (1-z)^n \\
&= \sum_{n=0}^{\infty} (-1)^n (z-1)^n \\
&= 1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots
\end{aligned}$$

for all z with $|z-1| < 1$.

Example 36.8. We have

$$\frac{1}{z^2+z^4} = \frac{1}{z^2} \cdot \frac{1}{1+z^2} = \frac{1}{z^2}\left(1 - z^2 + z^4 - z^6 + \cdots\right) = \frac{1}{z^2} - 1 + z^2 - z^4 + \cdots$$

for all z with $0 < |z| < 1$. Note that this representation is not a Maclaurin series, but is an example of a *Laurent series*, which we will consider next.