Lecture 33: The Maximum Modulus Principle

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Mathematics 39

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33.1 Maximum of the modulus

Lemma 33.1. Suppose f is analytic in the ϵ neighborhood U of z_0. If $|f(z)| \leq |f(z_0)|$ for all $z \in U$, then $f(z)$ is constant on U.

Proof. Let $0 < \rho < \epsilon$ and let C_ρ be the circle $|z - z_0| = \rho$. By the Cauchy integral formula, we know that

$$f(z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z - z_0} dz.$$

If we parametrize C_ρ by $z = z_0 + \rho e^{it}$, $0 \leq t \leq 2\pi$, then

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + \rho e^{it})}{\rho e^{it}} \cdot i\rho e^{it} dt = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{it}) dt.$$

(Note that this means that $f(z_0)$ is the average of the values of $f(z)$ on C_ρ.) Hence

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{it})| dt.$$

However, the assumption $|f(z_0)| \geq |f(z)|$ for all $z \in U$ implies that

$$\frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{it})| dt \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)| dt = |f(z_0)|.$$

Hence we must in fact have

$$|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \rho e^{it})| dt.$$

It follows that

$$\begin{aligned}
0 &= |f(z_0)| - \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{it})|dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|dt - \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{it})|dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} (|f(z_0)| - |f(z_0+\rho e^{it})|dt.
\end{aligned}$$

Since $|f(z_0)| - |f(z_0 + \rho e^{it})|$ is a continuous function of t and

$$|f(z_0)| - |f(z_0 + \rho e^{it})| \geq 0$$

for all $t \in [0, 2\pi]$, it follows that

$$|f(z_0)| = |f(z_0 + \rho e^{it})|$$

for all $t \in [0, 2\pi]$; that is, $|f(z_0)| = |f(z)|$ for all $z \in C_\rho$. Since ρ was arbitrary, it follows that $|f(z_0)| = |f(z)|$ for all $z \in U$. Finally, using a previous homework problem, we may now conclude that $f(z) = f(z_0)$ for all $z \in U$. □

With the lemma, we may now prove the *maximum modulus principle*.

Theorem 33.1. Suppose $D \subset \mathbb{C}$ is a domain and $f : D \to \mathbb{C}$ is analytic in D. If f is not a constant function, then $|f(z)|$ does not attain a maximum on D.

Proof. Suppose, to the contrary, that there exists a point $z_0 \in D$ for which $|f(z_0)| \geq |f(z)|$ for all other points $z \in D$. We will show that f must then be a constant function. Let w be any other point in D and consider a polygonal path L from z_0 to w. If D is not the entire plane, let δ be the minimum distance from L to the boundary of D; if D is the entire plane, let $\delta = 1$. Consider a finite sequence of points z_0, z_1, z_2, $\ldots$, $z_n = w$ with $z_k \in L$ and $|z_k - z_{k-1}| < \delta$ for $k = 1, 2, \ldots, n$. For example, we might construct these points by moving a distance $\frac{\delta}{2}$ along L from one to the next. Let U_k be the

δ neighborhood of z_k, $k = 0, 1, 2, \ldots, n$. By the lemma, $f(z) = f(z_0)$ for all $z \in U_0$. Since $z_1 \in U_0$, $f(z_1) = f(z_0)$. Then $|f(z_1)|$ is the maximum value of $|f(z)|$ on U_1, and so $f(z) = f(z_0)$ for all $z \in U_1$. Since $z_2 \in U_1$, we then have $f(z_2) = f(z_0)$, from which it follows that $f(z) = f(z_0)$ for all $z \in U_2$. Continuing in this manner, we eventually reach $f(w) = f(z_n) = f(z_0)$. Since w was an arbitrary point in D, it follows that $f(z) = f(z_0)$ for all $z \in D$. □

Corollary 33.1. Suppose $R \subset \mathbb{C}$ is a closed bounded region. If $f : R \to \mathbb{C}$ is continuous on R, analytic on the interior of R, and not constant, then the maximum value of $|f(z)|$ is attained at a point (or points) on the boundary of R and never at points in the interior of R. Moreover, if we write

$$f(x + iy) = u(x, y) + iv(x, y),$$

then the maximum value of $u(x, y)$ is attained at a point (or points) on the boundary of R and never at points in the interior of R.

Proof. The first part follows from the fact that a continuous function on a closed bounded set attains a maximum value, and from the maximum modulus principle this value cannot be attained in the interior of R. The second part follows from the observation that the modulus of the function

$$g(z) = e^{f(z)}$$

is

$$|g(z)| = e^{u(x,y)}.$$

□